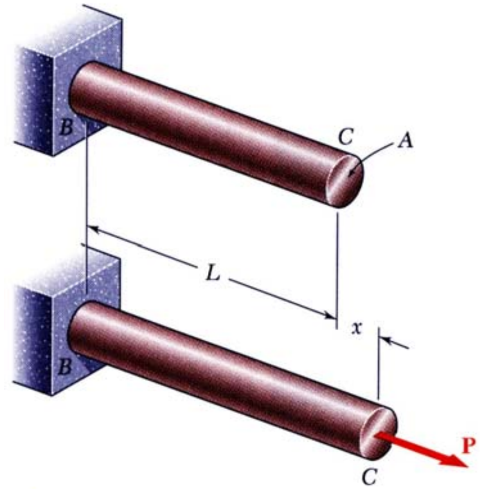
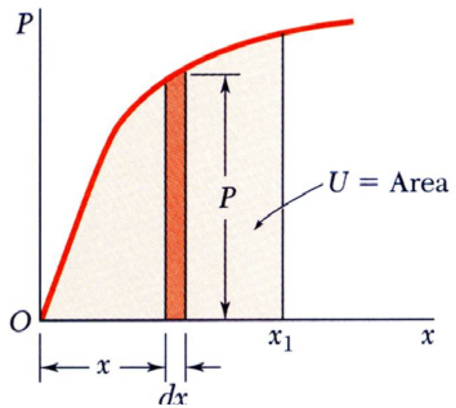
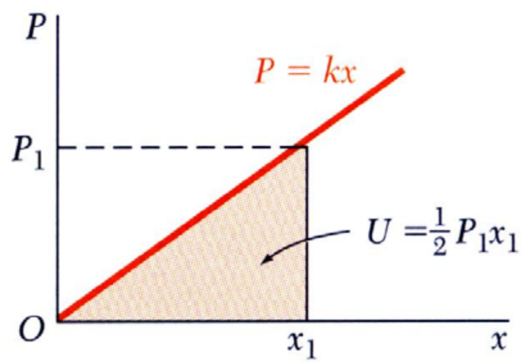
روش های انرژی در تحلیل سازه ها

Energy Methods

Strain Energy







- A uniform rod is subjected to a slowly increasing load
- The *elementary work* done by the load P as the rod elongates by a small *dx* is

$$dU = P\,dx = elementary\ work$$

which is equal to the area of width *dx* under the load-deformation diagram.

- The *total work* done by the load for a deformation x_1,

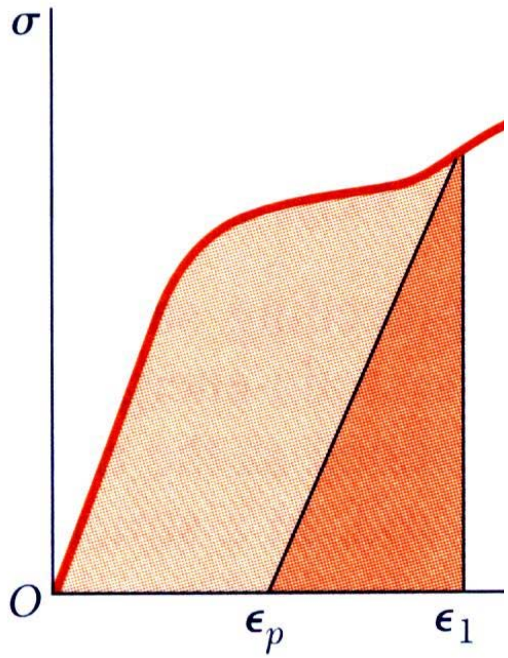
$$U = \int_0^{x_1} P\,dx = total\ work = strain\ energy$$

which results in an increase of *strain energy* in the rod.

- In the case of a linear elastic deformation,

$$U = \int_0^{x_1} kx\,dx = \tfrac{1}{2} k x_1^2 = \tfrac{1}{2} P_1 x_1$$

Strain Energy Density



- To eliminate the effects of size, evaluate the strain-energy per unit volume,

$$\frac{U}{V} = \int_0^{x_1} \frac{P}{A} \frac{dx}{L}$$

$$u = \int_0^{\varepsilon_1} \sigma_x \, d\varepsilon = \textit{strain energy density}$$

- The total strain energy density resulting from the deformation is equal to the area under the curve to ε_1.

- As the material is unloaded, the stress returns to zero but there is a permanent deformation. Only the strain energy represented by the triangular area is recovered.

- Remainder of the energy spent in deforming the material is dissipated as heat.

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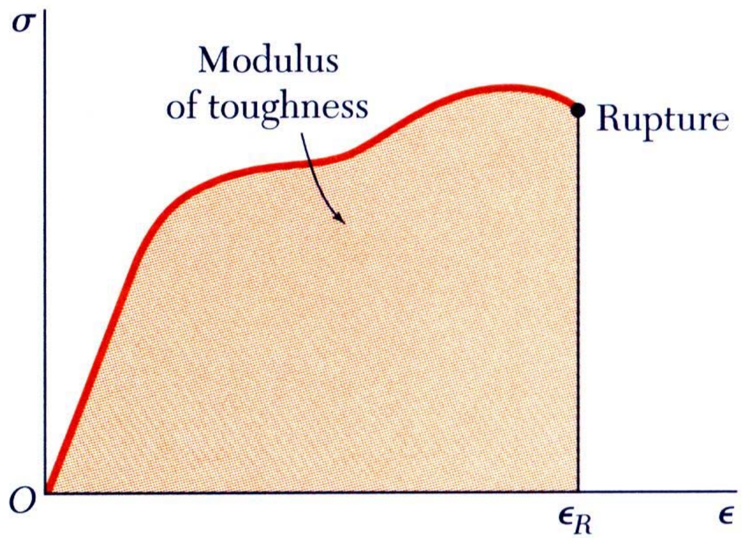
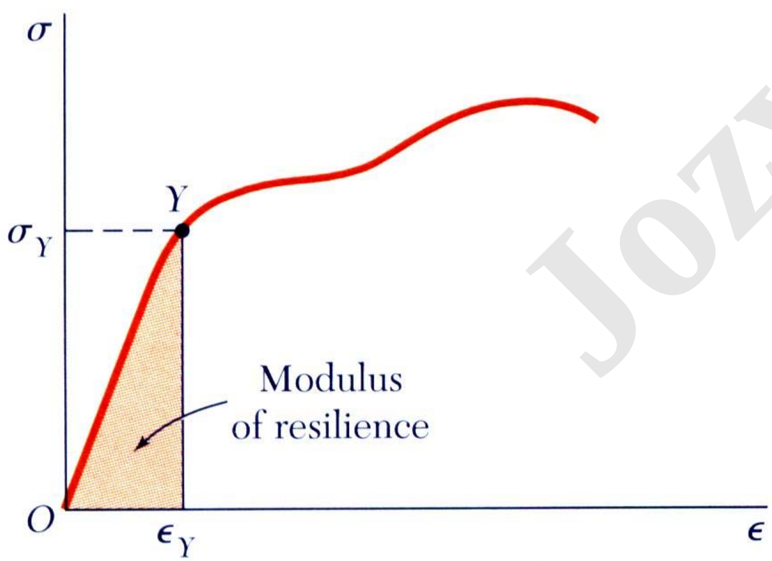
Strain-Energy Density





- The strain energy density resulting from setting $\varepsilon_1 = \varepsilon_R$ is the *modulus of toughness*.

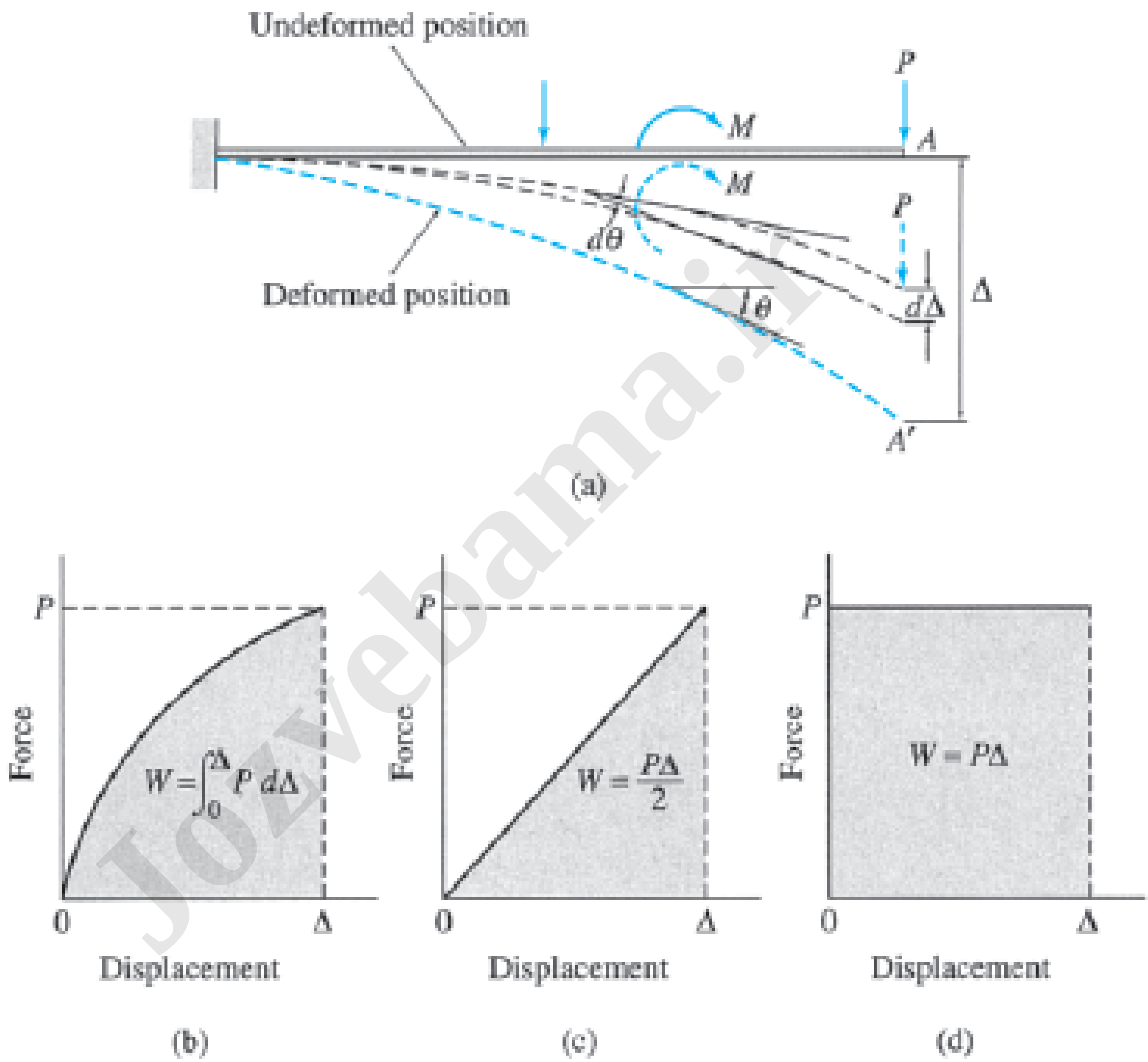
- The energy per unit volume required to cause the material to rupture is related to its ductility as well as its ultimate strength.

- If the stress remains within the proportional limit,

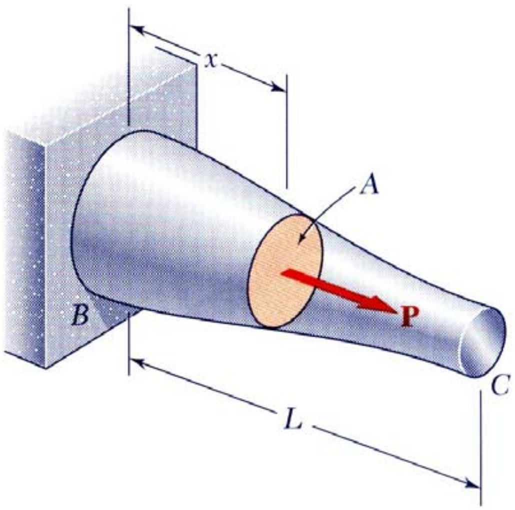
$$u = \int_0^{\varepsilon_1} E\varepsilon_1 \, d\varepsilon_x = \frac{E\varepsilon_1^2}{2} = \frac{\sigma_1^2}{2E}$$

- The strain energy density resulting from setting $\sigma_1 = \sigma_Y$ is the *modulus of resilience*.

$$u_Y = \frac{\sigma_Y^2}{2E} = \textit{modulus of resilience}$$


Undeformed position
P
M
A
M
P
$d\theta$
$d\Delta$
Δ
θ
Deformed position
A′
(a)
P
Force
$W = \int_0^{\Delta} P \, d\Delta$
0
Δ
Displacement
(b)
P
Force
$W = \frac{P\Delta}{2}$
0
Δ
Displacement
(c)
P
Force
$W = P\Delta$
0
Δ
Displacement
(d)

Elastic Strain Energy for Normal Stresses



- In an element with a nonuniform stress distribution,

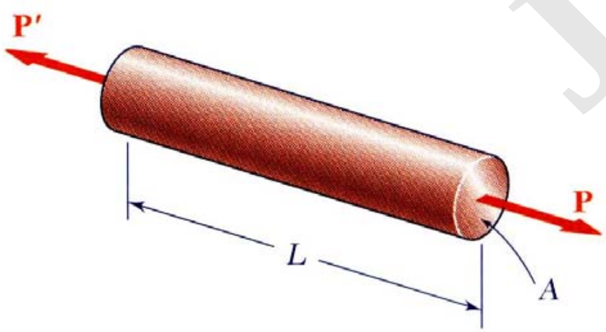
$$u = \lim_{\Delta V \to 0} \frac{\Delta U}{\Delta V} = \frac{dU}{dV} \qquad U = \int u \, dV = \text{total strain energy}$$

- For values of $u < u_Y$, i.e., below the proportional limit,

$$U = \int \frac{\sigma_x^2}{2E} dV = \textit{elastic strain energy}$$

- Under axial loading, $\sigma_x = P/A \qquad dV = A\,dx$

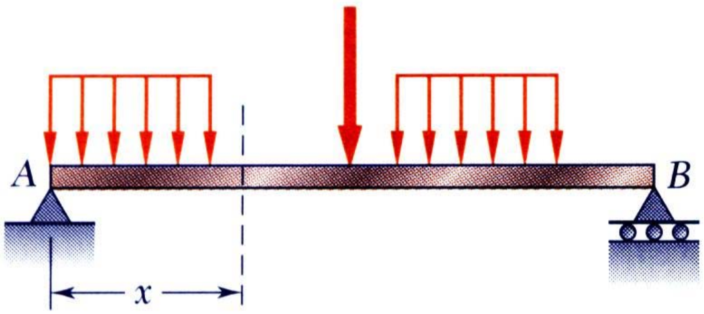
$$U = \int_0^L \frac{P^2}{2AE} dx$$



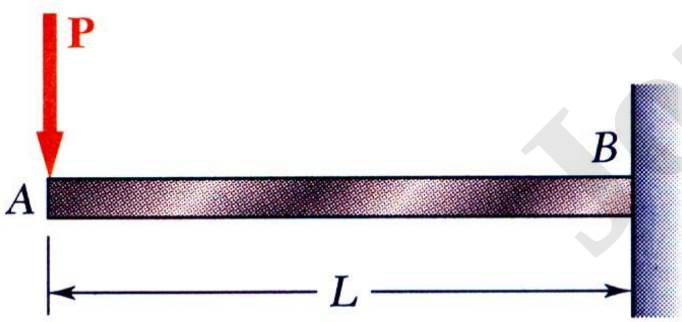
- For a rod of uniform cross-section,

$$U = \frac{P^2 L}{2AE}$$

Elastic Strain Energy for Normal Stresses



$$\sigma_x = \frac{M\,y}{I}$$



- For a beam subjected to a bending load,

$$U = \int \frac{\sigma_x^2}{2E} dV = \int \frac{M^2 y^2}{2EI^2} dV$$

- Setting $dV = dA\,dx$,

$$U = \int_0^L \int_A \frac{M^2 y^2}{2EI^2} dA\,dx = \int_0^L \frac{M^2}{2EI^2} \left(\int_A y^2 dA \right) dx$$

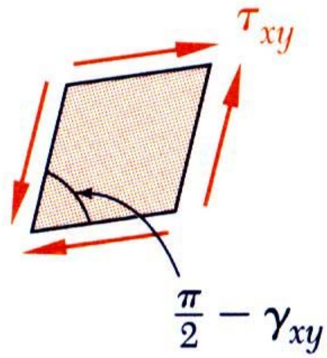
$$= \int_0^L \frac{M^2}{2EI} dx$$

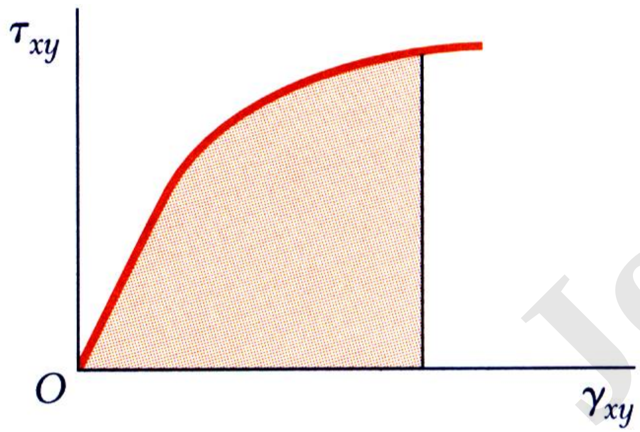
- For an end-loaded cantilever beam,

$$M = -Px$$

$$U = \int_0^L \frac{P^2 x^2}{2EI} dx = \frac{P^2 L^3}{6EI}$$

Strain Energy For Shearing Stresses





- For a material subjected to plane shearing stresses,

$$u = \int_0^{\gamma_{xy}} \tau_{xy}\, d\gamma_{xy}$$

- For values of τ_{xy} within the proportional limit,

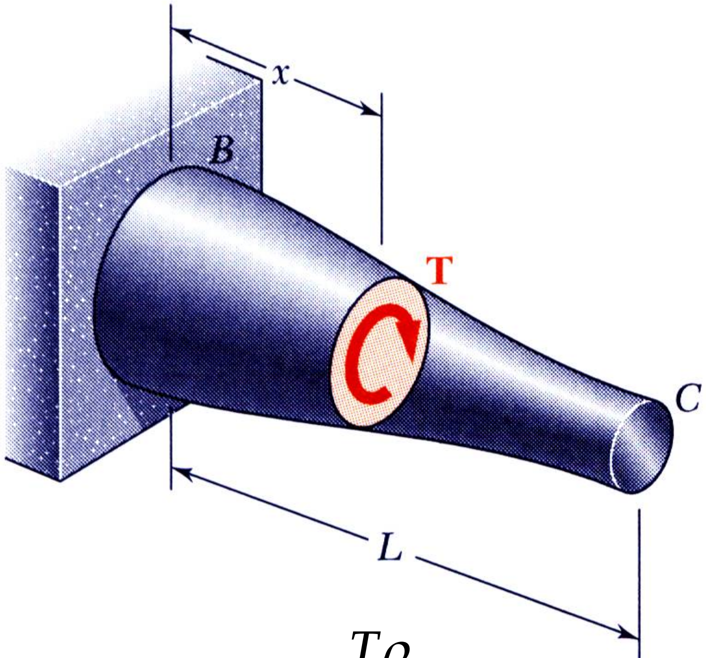
$$u = \tfrac{1}{2} G \gamma_{xy}^2 = \tfrac{1}{2} \tau_{xy} \gamma_{xy} = \frac{\tau_{xy}^2}{2G}$$

- The total strain energy is found from

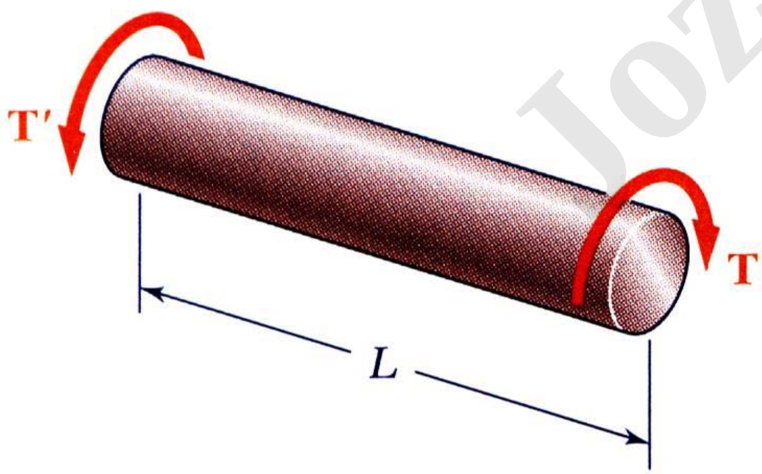
$$U = \int u\, dV$$

$$= \int \frac{\tau_{xy}^2}{2G} dV$$

Strain Energy For Shearing Stresses



$$\tau_{xy} = \frac{T\rho}{J}$$



- For a shaft subjected to a torsional load,

$$U = \int \frac{\tau_{xy}^2}{2G} dV = \int \frac{T^2 \rho^2}{2GJ^2} dV$$

- Setting $dV = dA\, dx$,

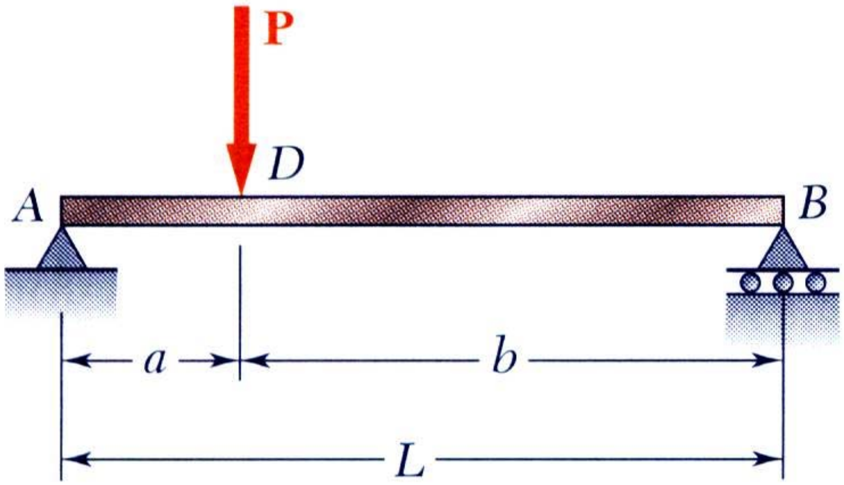
$$U = \int_0^L \int_A \frac{T^2 \rho^2}{2GJ^2} dA\, dx = \int_0^L \frac{T^2}{2GJ^2} \left(\int_A \rho^2 dA \right) dx$$

$$= \int_0^L \frac{T^2}{2GJ} dx$$

- In the case of a uniform shaft,

$$U = \frac{T^2 L}{2GJ}$$

Sample Problem 11.2



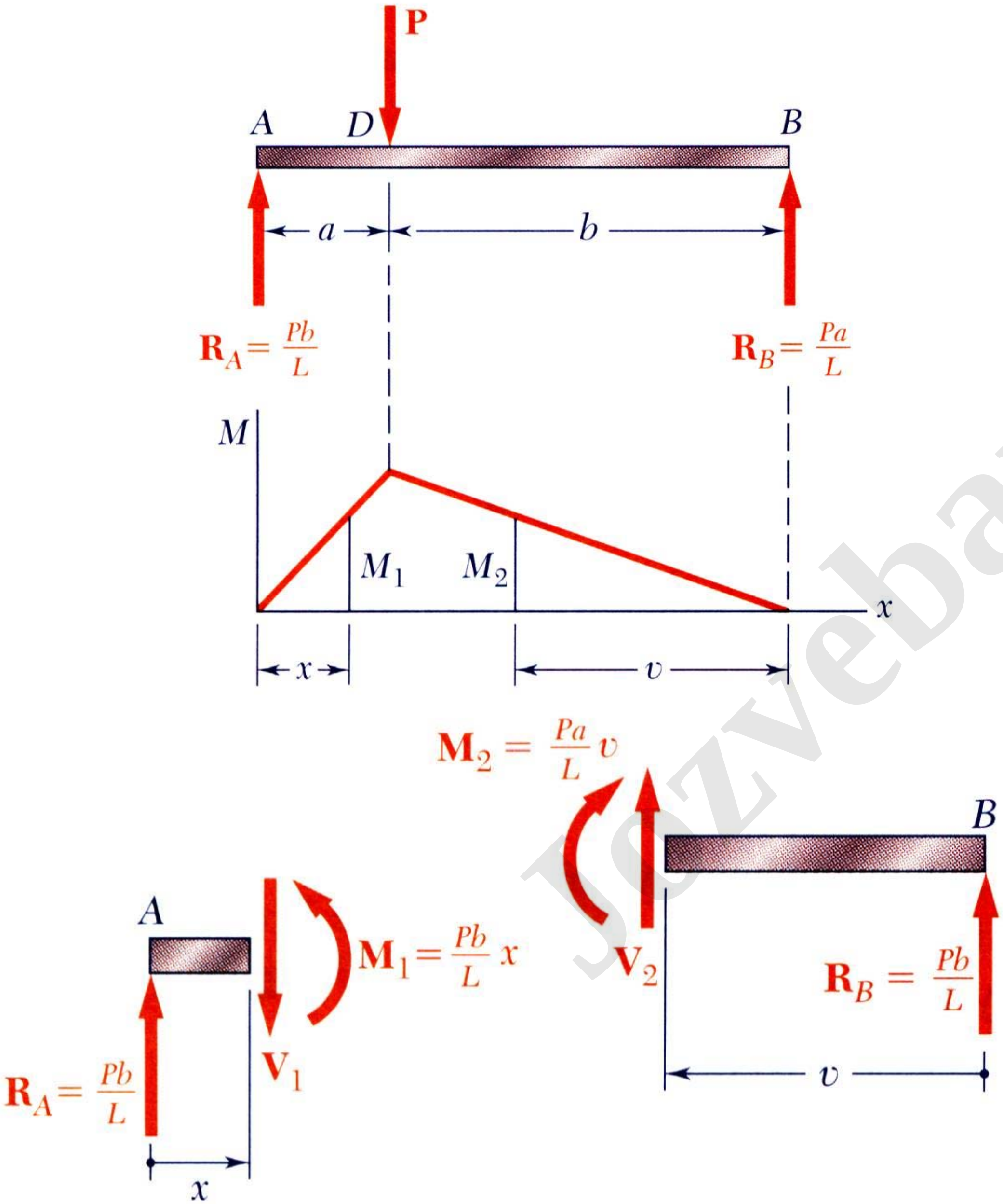
a) Taking into account only the normal stresses due to bending, determine the strain energy of the beam for the loading shown.

b) Evaluate the strain energy knowing that the beam is a W10x45, $P = 40$ kips, $L = 12$ ft, $a = 3$ ft, $b = 9$ ft, and $E = 29\text{x}10^6$ psi.

SOLUTION:

- Determine the reactions at A and B from a free-body diagram of the complete beam.
- Develop a diagram of the bending moment distribution.
- Integrate over the volume of the beam to find the strain energy.
- Apply the particular given conditions to evaluate the strain energy.

Sample Problem 11.2



SOLUTION:

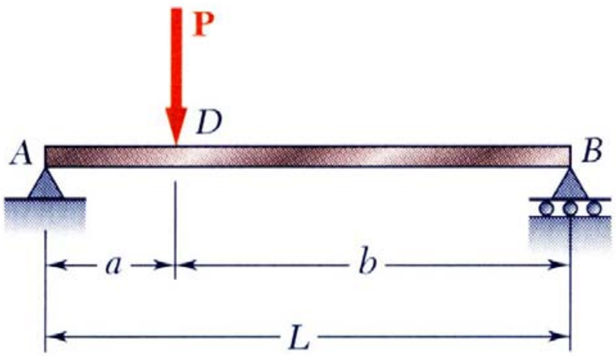
- Determine the reactions at A and B from a free-body diagram of the complete beam.

$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$

- Develop a diagram of the bending moment distribution.

$$M_1 = \frac{Pb}{L} x \qquad M_2 = \frac{Pa}{L} v$$

Sample Problem 11.2



Over the portion AD,

$$M_1 = \frac{Pb}{L}x$$

Over the portion BD,

$$M_2 = \frac{Pa}{L}v$$

$P = 45\,\text{kips}$ $\qquad L = 144\text{ in.}$

$a = 36\text{ in.}$ $\qquad b = 108\text{ in.}$

$E = 29\times10^{3}\,\text{ksi}$ $\qquad I = 248\text{ in}^{4}$

- Integrate over the volume of the beam to find the strain energy.

$$U = \int_0^a \frac{M_1^2}{2EI}dx + \int_0^b \frac{M_2^2}{2EI}dv$$

$$= \frac{1}{2EI}\int_0^a \left(\frac{Pb}{L}x\right)^2 dx + \frac{1}{2EI}\int_0^b \left(\frac{Pa}{L}x\right)^2 dx$$

$$= \frac{1}{2EI}\frac{P^2}{L^2}\left(\frac{b^2a^3}{3} + \frac{a^2b^3}{3}\right) = \frac{P^2a^2b^2}{6EIL^2}(a+b)$$

$$U = \frac{P^2a^2b^2}{6EIL}$$

$$U = \frac{(40\,\text{kips})^2(36\text{ in})^2(108\text{ in})^2}{6\left(29\times10^3\text{ ksi}\right)\left(248\text{ in}^4\right)(144\text{ in})}$$

$$U = 3.89\text{ in}\cdot\text{kips}$$

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Strain Energy for a General State of Stress

- Previously found strain energy due to uniaxial stress and plane shearing stress. For a general state of stress,

$$u = \tfrac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right)$$

- With respect to the principal axes for an elastic, isotropic body,

$$u = \frac{1}{2E}\left[\sigma_a^2 + \sigma_b^2 + \sigma_c^2 - 2v\left(\sigma_a\sigma_b + \sigma_b\sigma_c + \sigma_c\sigma_a\right)\right]$$

$$= u_v + u_d$$

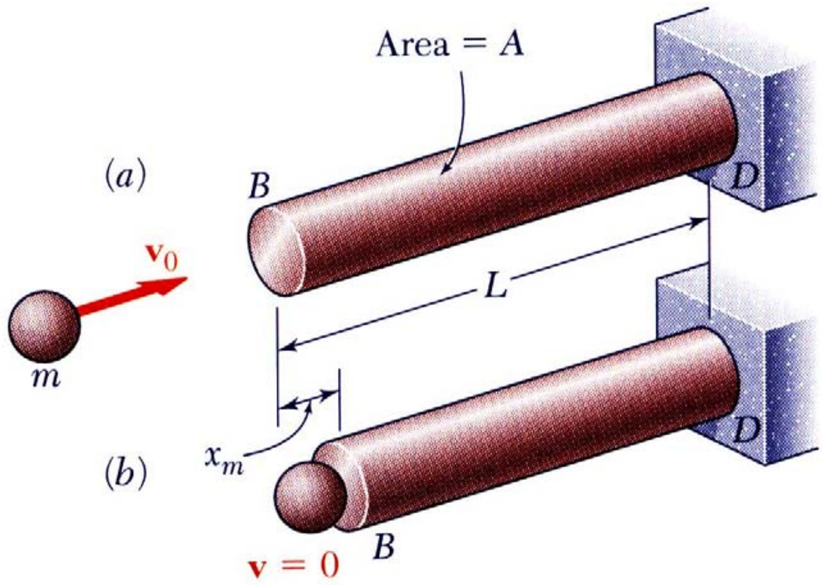
$$u_v = \frac{1-2v}{6E}\left(\sigma_a + \sigma_b + \sigma_c\right)^2 = \text{due to volume change}$$

$$u_d = \frac{1}{12G}\left[\left(\sigma_a - \sigma_b\right)^2 + \left(\sigma_b - \sigma_c\right)^2 + \left(\sigma_c - \sigma_a\right)^2\right] = \text{due to distortion}$$

- Basis for the *maximum distortion energy* failure criteria,

$$u_d < \left(u_d\right)_Y = \frac{\sigma_Y^2}{6G} \quad \text{for a tensile test specimen}$$

Impact Loading



- Consider a rod which is hit at its end with a body of mass *m* moving with a velocity v_0.

- Rod deforms under impact. Stresses reach a maximum value σ_m and then disappear.

- To determine the maximum stress σ_m
  - Assume that the kinetic energy is transferred entirely to the structure,

$$U_m = \frac{1}{2} m v_0^2$$

  - Assume that the stress-strain diagram obtained from a static test is also valid under impact loading.

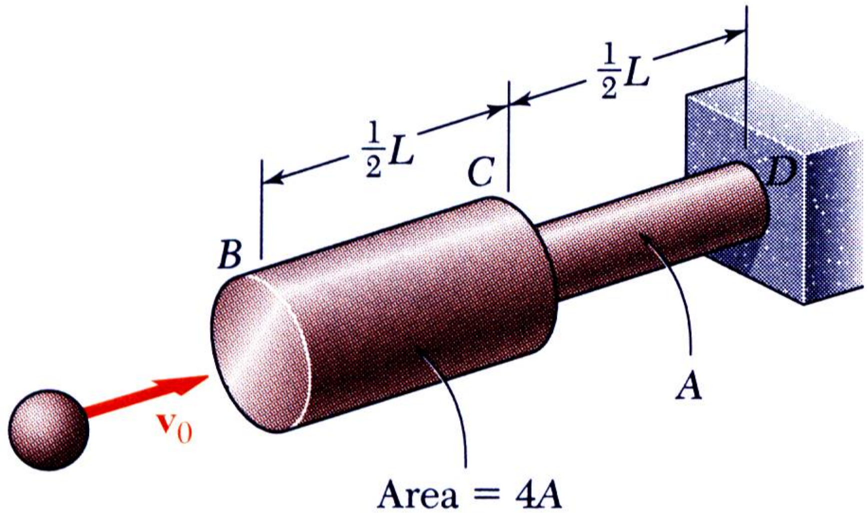
- Maximum value of the strain energy,

$$U_m = \int \frac{\sigma_m^2}{2E} dV$$

- For the case of a uniform rod,

$$\sigma_m = \sqrt{\frac{2U_m E}{V}} = \sqrt{\frac{m v_0^2 E}{V}}$$

Example 11.06

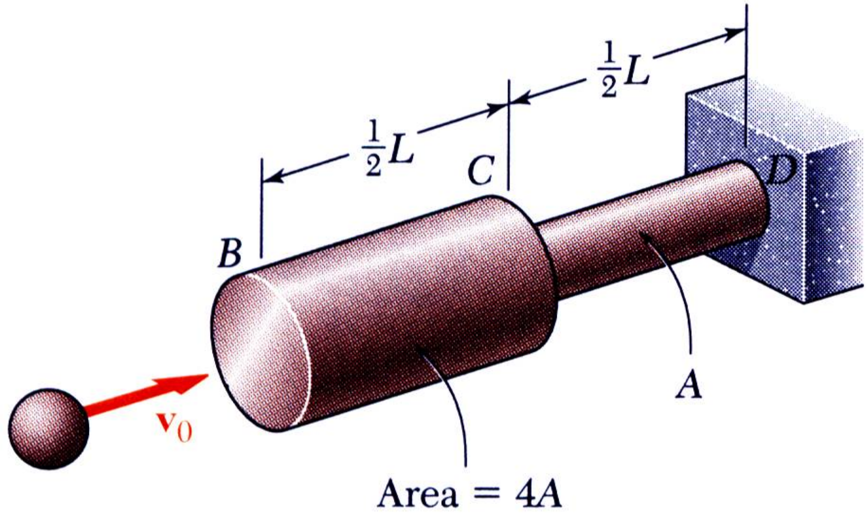


Body of mass m with velocity v_0 hits the end of the nonuniform rod BCD. Knowing that the diameter of the portion BC is twice the diameter of portion CD, determine the maximum value of the normal stress in the rod.

SOLUTION:

- Due to the change in diameter, the normal stress distribution is nonuniform.
- Find the static load P_m which produces the same strain energy as the impact.
- Evaluate the maximum stress resulting from the static load P_m

Example 11.06



SOLUTION:

- Due to the change in diameter, the normal stress distribution is nonuniform.

$$U_m = \tfrac{1}{2} m v_0^2$$

$$= \int \frac{\sigma_m^2}{2E}\, dV \neq \frac{\sigma_m^2 V}{2E}$$

- Find the static load P_m which produces the same strain energy as the impact.

$$U_m = \frac{P_m^2 (L/2)}{AE} + \frac{P_m^2 (L/2)}{4AE} = \frac{5}{16}\frac{P_m^2 L}{AE}$$

$$P_m = \sqrt{\frac{16}{5}\frac{U_m AE}{L}}$$

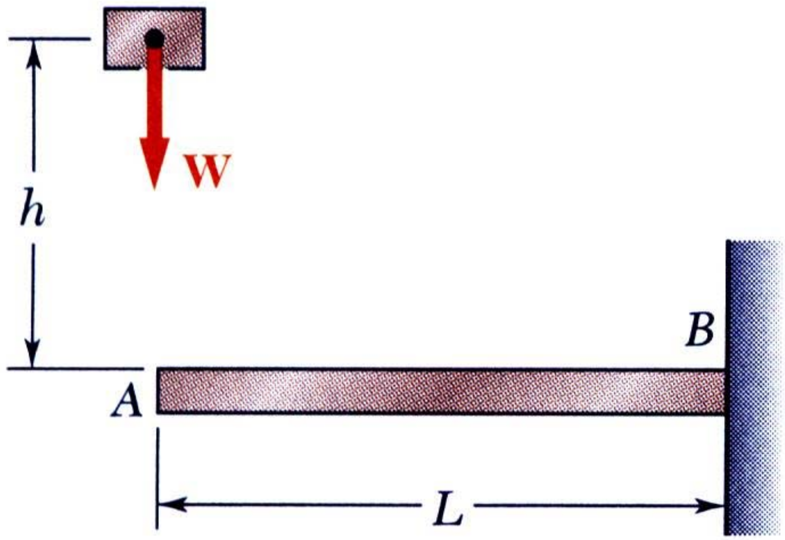
- Evaluate the maximum stress resulting from the static load P_m

$$\sigma_m = \frac{P_m}{A}$$

$$= \sqrt{\frac{16}{5}\frac{U_m E}{AL}}$$

$$= \sqrt{\frac{8}{5}\frac{m v_0^2 E}{AL}}$$

Example 11.07

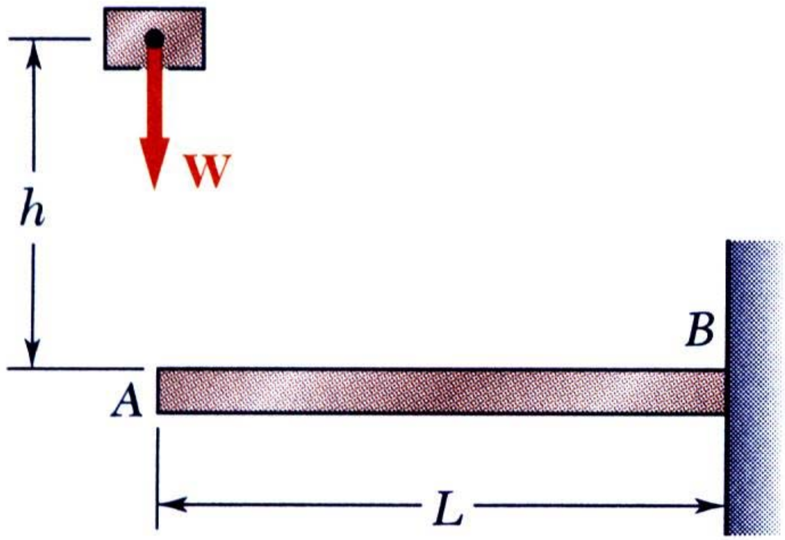


A block of weight W is dropped from a height h onto the free end of the cantilever beam. Determine the maximum value of the stresses in the beam.

SOLUTION:

- The normal stress varies linearly along the length of the beam as across a transverse section.
- Find the static load P_m which produces the same strain energy as the impact.
- Evaluate the maximum stress resulting from the static load P_m

Example 11.07



SOLUTION:

- The normal stress varies linearly along the length of the beam as across a transverse section.

$$U_m = Wh$$

$$= \int \frac{\sigma_m^2}{2E} dV \neq \frac{\sigma_m^2 V}{2E}$$

- Find the static load P_m which produces the same strain energy as the impact.

For an end-loaded cantilever beam,

$$U_m = \frac{P_m^2 L^3}{6EI}$$

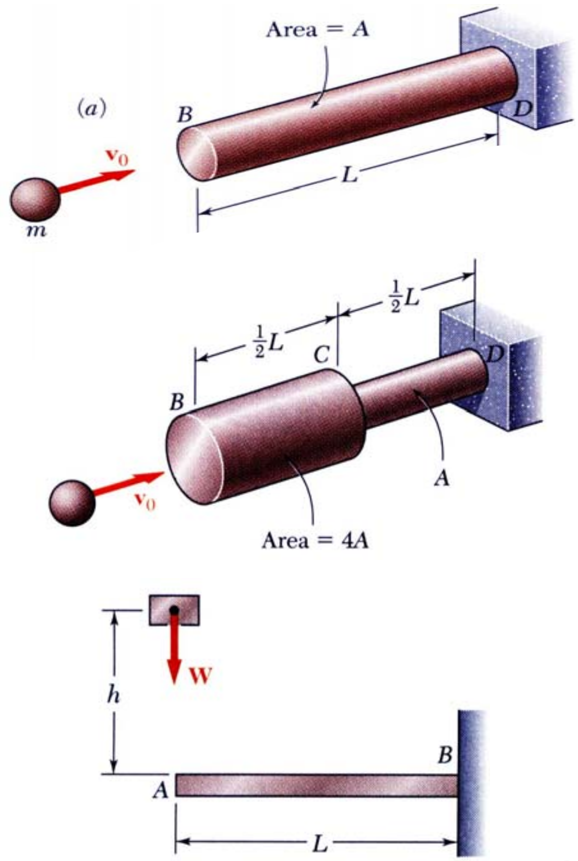
$$P_m = \sqrt{\frac{6U_m EI}{L^3}}$$

- Evaluate the maximum stress resulting from the static load P_m

$$\sigma_m = \frac{|M|_m c}{I} = \frac{P_m L c}{I}$$

$$= \sqrt{\frac{6U_m E}{L(I/c^2)}} = \sqrt{\frac{6WhE}{L(I/c^2)}}$$

Design for Impact Loads



Maximum stress reduced by:

- uniformity of stress
- low modulus of elasticity with high yield strength
- high volume

- For the case of a uniform rod,

$$\sigma_m = \sqrt{\frac{2U_m E}{V}}$$

- For the case of the nonuniform rod,

$$\sigma_m = \sqrt{\frac{16}{5}\frac{U_m E}{AL}}$$

$$V = 4A(L/2) + A(L/2) = 5AL/2$$

$$\sigma_m = \sqrt{\frac{8U_m E}{V}}$$

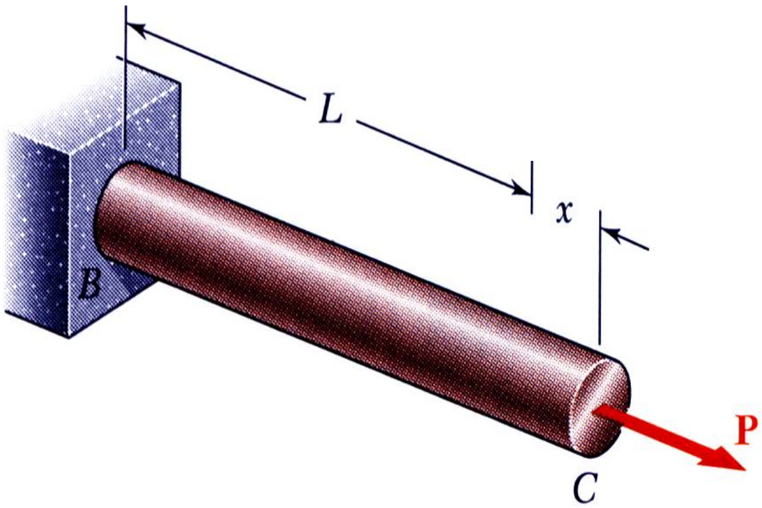
- For the case of the cantilever beam

$$\sigma_m = \sqrt{\frac{6U_m E}{L\left(I/c^2\right)}}$$

$$L\left(I/c^2\right) = L\left(\tfrac{1}{4}\pi c^4/c^2\right) = \tfrac{1}{4}\left(\pi c^2 L\right) = \tfrac{1}{4}V$$

$$\sigma_m = \sqrt{\frac{24U_m E}{V}}$$

Work and Energy Under a Single Load



- Previously, we found the strain energy by integrating the energy density over the volume.
  For a uniform rod,

$$U = \int u\,dV = \int \frac{\sigma^2}{2E}\,dV$$

$$= \int_0^L \frac{(P_1/A)^2}{2E} A\,dx = \frac{P_1^2 L}{2AE}$$

- Strain energy may also be found from the work of the single load P_1,

$$U = \int_0^{x_1} P\,dx$$

- For an elastic deformation,

$$U = \int_0^{x_1} P\,dx = \int_0^{x_1} kx\,dx = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}P_1 x_1$$

- Knowing the relationship between force and displacement,

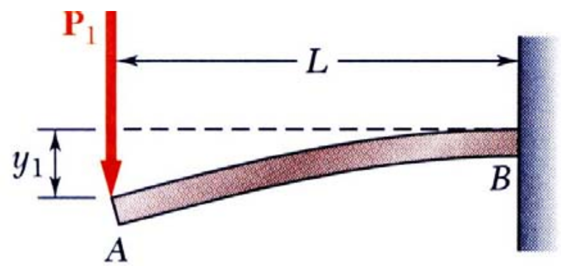
$$x_1 = \frac{P_1 L}{AE}$$

$$U = \tfrac{1}{2}P_1\left(\frac{P_1 L}{AE}\right) = \frac{P_1^2 L}{2AE}$$

Work and Energy Under a Single Load

- Strain energy may be found from the work of other types of single concentrated loads.

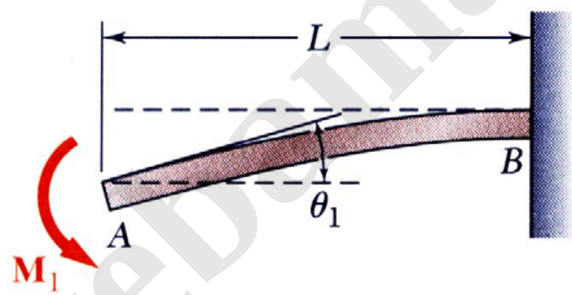
- Transverse load



$$U = \int_0^{y_1} P\,dy = \tfrac{1}{2} P_1 y_1$$

$$= \tfrac{1}{2} P_1 \left(\frac{P_1 L^3}{3EI} \right) = \frac{P_1^2 L^3}{6EI}$$

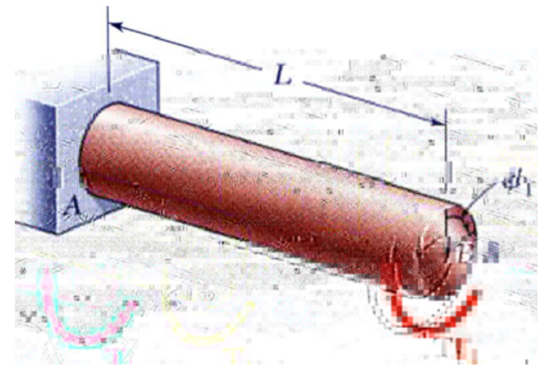
- Bending couple



$$U = \int_0^{\theta_1} M\,d\theta = \tfrac{1}{2} M_1 \theta_1$$

$$= \tfrac{1}{2} M_1 \left(\frac{M_1 L}{EI} \right) = \frac{M_1^2 L}{2EI}$$

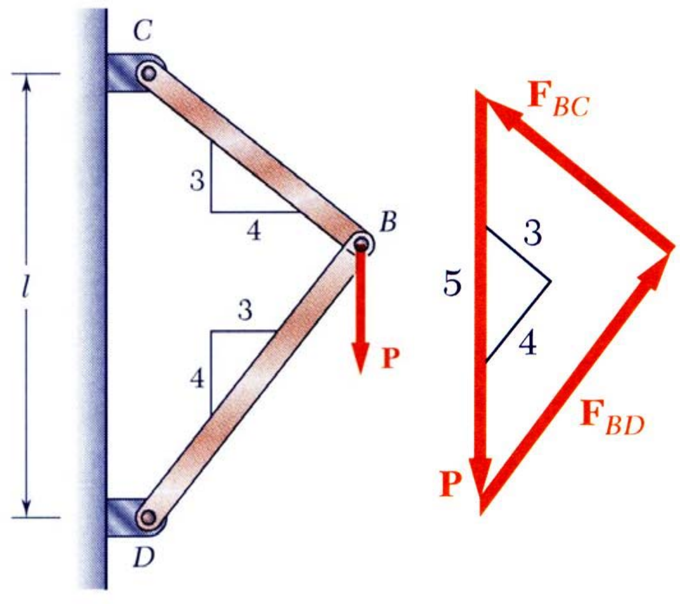
- Torsional couple



$$U = \int_0^{\phi_1} T\,d\phi = \tfrac{1}{2} T_1 \phi_1$$

$$= \tfrac{1}{2} T_1 \left(\frac{T_1 L}{JG} \right) = \frac{T_1^2 L}{2JG}$$

Deflection Under a Single Load



From the given geometry,

$$L_{BC} = 0.6l \quad L_{BD} = 0.8l$$

From statics,

$$F_{BC} = +0.6P \quad F_{BD} = -0.8P$$

- If the strain energy of a structure due to a single concentrated load is known, then the equality between the work of the load and energy may be used to find the deflection.

- Strain energy of the structure,

$$U = \frac{F_{BC}^2 L_{BC}}{2AE} + \frac{F_{BD}^2 L_{BD}}{2AE}$$

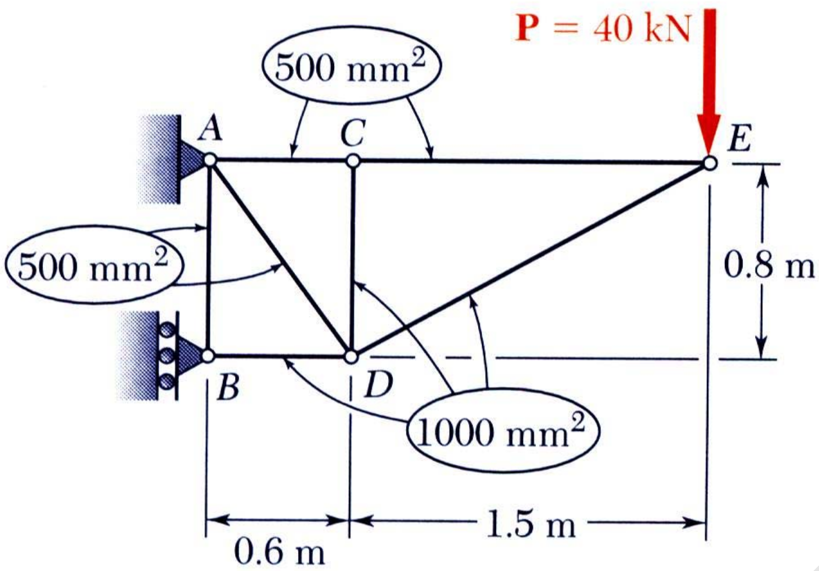
$$= \frac{P^2 l\left[(0.6)^3 + (0.8)^3\right]}{2AE} = 0.364\frac{P^2 l}{AE}$$

- Equating work and strain energy,

$$U = 0.364\frac{P^2 L}{AE} = \tfrac{1}{2}P y_B$$

$$y_B = 0.728\frac{Pl}{AE}$$

Sample Problem 11.4

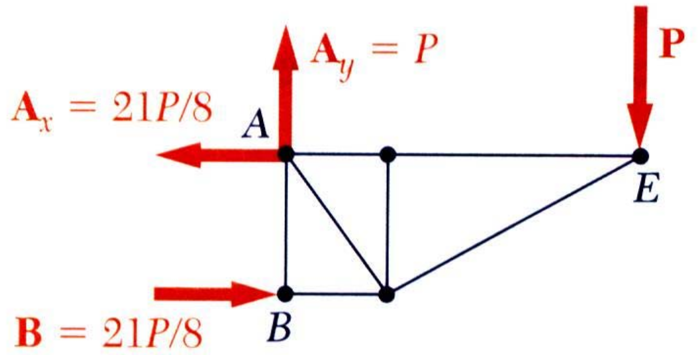


Members of the truss shown consist of sections of aluminum pipe with the cross-sectional areas indicated. Using $E = 73$ GPa, determine the vertical deflection of the point *E* caused by the load P.

SOLUTION:

- Find the reactions at A and B from a free-body diagram of the entire truss.
- Apply the method of joints to determine the axial force in each member.
- Evaluate the strain energy of the truss due to the load *P*.
- Equate the strain energy to the work of *P* and solve for the displacement.

Sample Problem 11.4

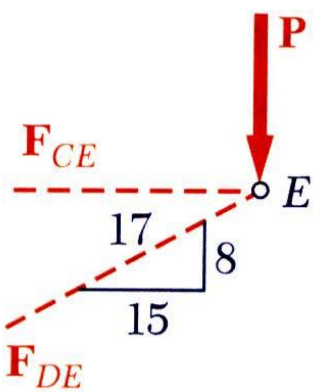


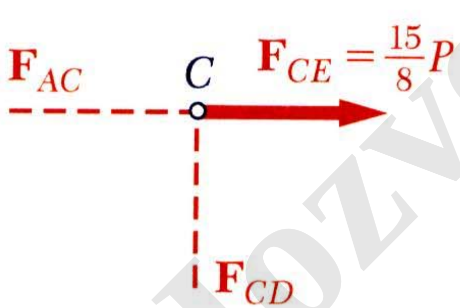
SOLUTION:

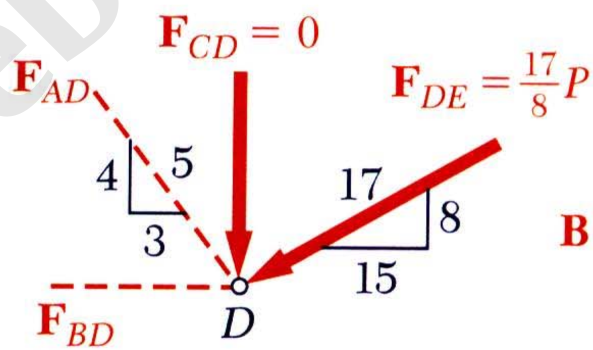
- Find the reactions at A and B from a free-body diagram of the entire truss.

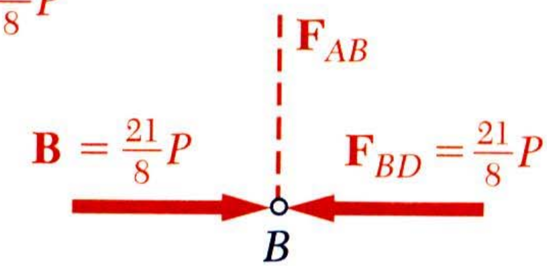
$$A_x = -21P/8 \qquad A_y = P \qquad B = 21P/8$$

- Apply the method of joints to determine the axial force in each member.





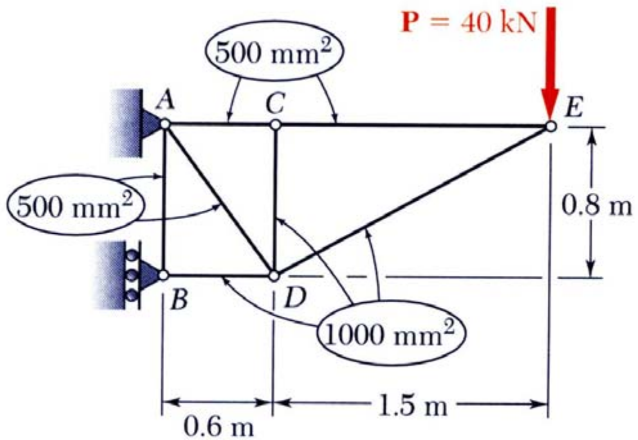




$$F_{DE} = -\frac{17}{8}P \qquad F_{AC} = +\frac{15}{8}P \qquad F_{DE} = \frac{5}{4}P \qquad F_{AB} = 0$$

$$F_{CE} = +\frac{15}{8}P \qquad F_{CD} = 0 \qquad F_{CE} = -\frac{21}{8}P$$

Sample Problem 11.4



Member	F_i	L_i, m	A_i, m²	$\frac{F_i^2 L_i}{A_i}$
AB	0	0.8	500×10^{-6}	0
AC	$+15P/8$	0.6	500×10^{-6}	$4\,219P^2$
AD	$+5P/4$	1.0	500×10^{-6}	$3\,125P^2$
BD	$-21P/8$	0.6	1000×10^{-6}	$4\,134P^2$
CD	0	0.8	1000×10^{-6}	0
CE	$+15P/8$	1.5	500×10^{-6}	$10\,547P^2$
DE	$-17P/8$	1.7	1000×10^{-6}	$7\,677P^2$

- Evaluate the strain energy of the truss due to the load *P*.

$$U = \sum \frac{F_i^2 L_i}{2 A_i E} = \frac{1}{2E} \sum \frac{F_i^2 L_i}{A_i}$$

$$= \frac{1}{2E}\left(29700 P^2\right)$$

- Equate the strain energy to the work by *P* and solve for the displacement.

$$\tfrac{1}{2} P y_E = U$$

$$y_E = \frac{2U}{P} = \frac{2}{P}\left(\frac{29700 P^2}{2E}\right)$$

$$y_E = \frac{\left(29.7 \times 10^3\right)\left(40 \times 10^3\right)}{73 \times 10^9}$$

$$y_E = 16.27\,\text{mm} \downarrow$$

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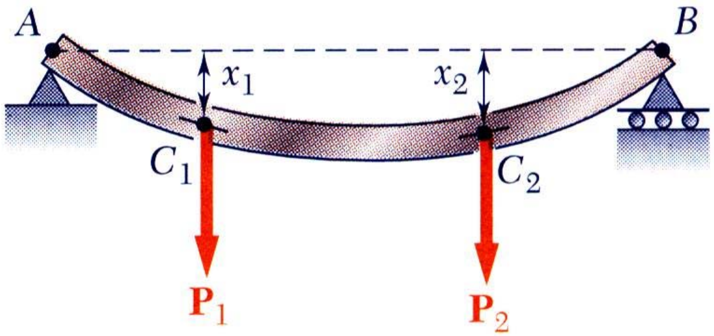
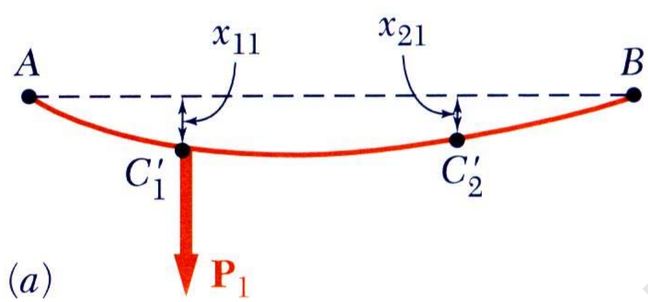
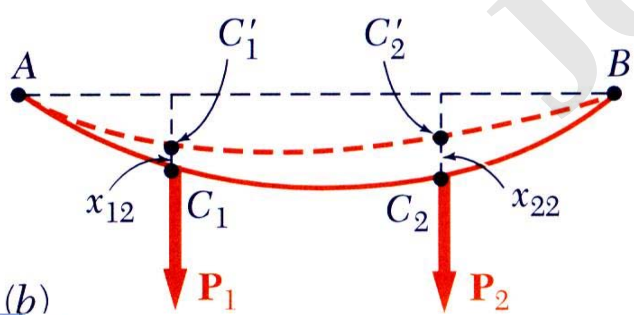
Work and Energy Under Several Loads







- Deflections of an elastic beam subjected to two concentrated loads,

$$x_1 = x_{11} + x_{12} = \alpha_{11}P_1 + \alpha_{12}P_2$$

$$x_2 = x_{21} + x_{22} = \alpha_{21}P_1 + \alpha_{22}P_2$$

- Compute the strain energy in the beam by evaluating the work done by slowly applying P_1 followed by P_2,

$$U = \tfrac{1}{2}\left(\alpha_{11}P_1^2 + 2\alpha_{12}P_1P_2 + \alpha_{22}P_2^2\right)$$

- Reversing the application sequence yields

$$U = \tfrac{1}{2}\left(\alpha_{22}P_2^2 + 2\alpha_{21}P_2P_1 + \alpha_{11}P_1^2\right)$$

- Strain energy expressions must be equivalent. It follows that $\alpha_{12}=\alpha_{21}$ (*Maxwell's reciprocal theorem*).

7.2 Principle of Virtual Work

The *principle of virtual work*, which was introduced by John Bernoulli in 1717, provides a powerful analytical tool for many problems of structural mechanics. In this section, we study two formulations of this principle, namely, the *principle of virtual displacements for rigid bodies* and the *principle of virtual forces for deformable bodies*. The latter formulation is used in the following sections to develop the *method of virtual work*, which is considered to be one of the most general methods for determining deflections of structures.

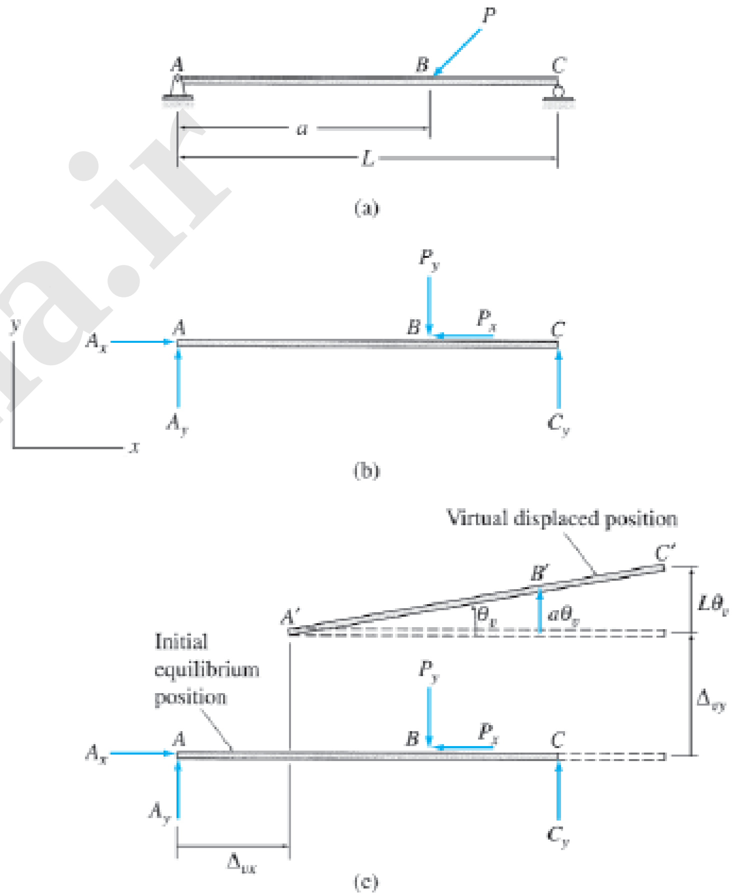
Principle of Virtual Displacements for Rigid Bodies

The principle of virtual displacements for rigid bodies can be stated as follows:

> If a rigid body is in equilibrium under a system of forces and if it is subjected to any small virtual rigid-body displacement, the virtual work done by the external forces is zero.

The term *virtual* simply means imaginary, not real. Consider the beam shown in Fig. 7.2(a). The free-body diagram of the beam is shown in Fig. 7.2(b), in which P_x and P_y represent the components of the external load P in the x and y directions, respectively.

Now, suppose that the beam is given an arbitrary small virtual rigid-body displacement from its initial equilibrium position ABC to another position $A'B'C'$, as shown in Fig. 7.2(c). As shown in this figure, the total virtual rigid-body displacement of the beam can be decomposed into translations Δ_{vx} and Δ_{vy} in the x and y directions, respectively, and a rotation θ_v about point A. Note that the subscript v is used here to identify the displacements as virtual quantities. As the beam undergoes the virtual displacement from position ABC to position $A'B'C'$, the forces acting on it perform work, which is called *virtual work*. The total



virtual work, W_{ve}, performed by the external forces acting on the beam can be expressed as the sum of the virtual work W_{vx} and W_{vy} done during translations in the x and y directions, respectively, and the virtual work W_{vr}, done during the rotation; that is,

$$W_{ve} = W_{vx} + W_{vy} + W_{vr} \tag{7.7}$$

During the virtual translations Δ_{vx} and Δ_{vy} of the beam, the virtual work done by the forces is given by

$$W_{vx} = A_x\Delta_{vx} - P_x\Delta_{vx} = (A_x - P_x)\Delta_{vx} = \left(\sum F_x\right)\Delta_{vx} \tag{7.8}$$

and

$$W_{vy} = A_y\Delta_{vy} - P_y\Delta_{vy} + C_y\Delta_{vy} = (A_y - P_y + C_y)\Delta_{vy} = \left(\sum F_y\right)\Delta_{vy} \tag{7.9}$$

(see Fig. 7.2(c)). The virtual work done by the forces during the small virtual rotation θ_v can be expressed as

$$W_{vr} = -P_y(a\theta_v) + C_y(L\theta_v) = (-aP_y + LC_y)\theta_v = \left(\sum M_A\right)\theta_v \tag{7.10}$$

By substituting Eqs. (7.8) through (7.10) into Eq. (7.7), we write the total virtual work done as

$$W_{ve} = \left(\sum F_x\right)\Delta_{vx} + \left(\sum F_y\right)\Delta_{vy} + \left(\sum M_A\right)\theta_v \tag{7.11}$$

Because the beam is in equilibrium, $\sum F_x = 0$, $\sum F_y = 0$, and $\sum M_A = 0$; therefore, Eq. (7.11) becomes

$$W_{ve} = 0 \tag{7.12}$$

which is the mathematical statement of the principle of virtual displacements for rigid bodies.

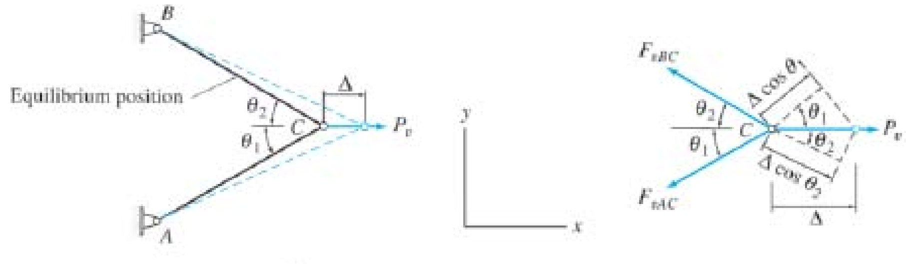
Principle of Virtual Forces for Deformable Bodies

The principle of virtual forces for deformable bodies can be stated as follows:

> If a deformable structure is in equilibrium under a virtual system of forces (and couples) and if it is subjected to any small real deformation consistent with the support and continuity conditions of the structure, then the virtual external work done by the virtual external forces (and couples) acting through the real external displacements (and rotations) is equal to the virtual internal work done by the virtual internal forces (and couples) acting through the real internal displacements (and rotations).

In this statement, the term *virtual* is associated with the forces to indicate that the force system is arbitrary and does not depend on the action causing the real deformation.

To demonstrate the validity of this principle, consider the two-member truss shown in Fig. 7.3(a). The truss is in equilibrium under the action of a virtual external force P_v as shown. The free-body diagram of joint C of the truss is shown in Fig. 7.3(b). Since joint C is in equilibrium,



(a) (b)

the virtual external and internal forces acting on it must satisfy the following two equilibrium equations:

$$\begin{aligned} \sum F_x = 0 \qquad & P_v - F_{vAC} \cos \theta_1 - F_{vBC} \cos \theta_2 = 0 \\ \sum F_y = 0 \qquad & -F_{vAC} \sin \theta_1 + F_{vBC} \sin \theta_2 = 0 \end{aligned} \tag{7.13}$$

in which F_{vAC} and F_{vBC} represent the virtual internal forces in members AC and BC, respectively, and θ_1 and θ_2 denote, respectively, the angles of inclination of these members with respect to the horizontal (Fig. 7.3(a)).

Now, let us assume that joint C of the truss is given a small real displacement, Δ, to the right from its equilibrium position, as shown in Fig. 7.3(a). Note that the deformation is consistent with the support conditions of the truss; that is, joints A and B, which are attached to supports, are not displaced. Because the virtual forces acting at joints A and B do not perform any work, the total virtual work for the truss (W_v) is equal to the algebraic sum of the work of the virtual forces acting at joint C; that is,

$$W_v = P_v \Delta - F_{vAC}(\Delta \cos \theta_1) - F_{vBC}(\Delta \cos \theta_2)$$

or

$$W_v = (P_v - F_{vAC} \cos \theta_1 - F_{vBC} \cos \theta_2)\Delta \tag{7.14}$$

As indicated by Eq. (7.13), the term in the parentheses on the right-hand side of Eq. (7.14) is zero; therefore, the total virtual work is $W_v = 0$. Thus, Eq. (7.14) can be expressed as

$$P_v \Delta = F_{vAC}(\Delta \cos \theta_1) + F_{vBC}(\Delta \cos \theta_2) \tag{7.15}$$

in which the quantity on the left-hand side represents the virtual external work (W_{ve}) done by the virtual external force, P_v, acting through the real external displacement, Δ. Also, realizing that the terms $\Delta \cos \theta_1$ and $\Delta \cos \theta_2$ are equal to the real internal displacements (elongations) of members AC and BC, respectively, we can conclude that the right-hand side of Eq. (7.15) represents the virtual internal work (W_{vi}) done by the virtual internal forces acting through the real internal displacements; that is

$$W_{ve} = W_{vi} \tag{7.16}$$

which is the mathematical statement of the principle of virtual forces for deformable bodies.

It should be realized that the principle of virtual forces as described here is applicable regardless of the cause of real deformations; that is, deformations due to loads, temperature changes, or any other effect can be determined by the application of the principle. However, the deformations must be small enough so that the virtual forces remain constant in magnitude and direction while performing the virtual work. Also, although the application of this principle in this text is limited to elastic structures, the principle is valid regardless of whether the structure is elastic or not.

The method of virtual work is based on the principle of virtual forces for deformable bodies as expressed by Eq. (7.16), which can be rewritten as

$$\text{virtual external work} = \text{virtual internal work} \tag{7.17}$$

or, more specifically, as

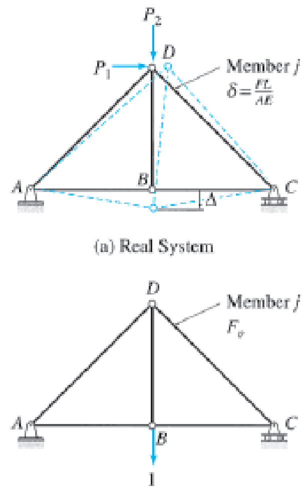
$$\sum \left(\begin{array}{l} \overbrace{\text{virtual external force}}^{\textit{Virtual system}} \times \\ \underbrace{\text{real external displacement}}_{\textit{Real system}} \end{array} \right) = \sum \left(\begin{array}{l} \text{virtual internal force} \times \\ \text{real internal displacement} \end{array} \right) \tag{7.18}$$

in which the terms *forces* and *displacements* are used in a general sense and include moments and rotations, respectively. Note that because the virtual forces are independent of the actions causing the real deformation and remain constant during the real deformation, the expressions of the external and internal virtual work in Eq. (7.18) do not contain the factor 1/2.

As Eq. (7.18) indicates, the method of virtual work employs two separate systems: a virtual force system and the real system of loads (or other effects) that cause the deformation to be determined. To determine the deflection (or slope) at any point of a structure, a virtual force system is selected so that the desired deflection (or rotation) will be the only unknown in Eq. (7.18). The explicit expressions of the virtual work method to be used for computing deflections of trusses, beams, and frames are developed in the following three sections.

7.3 Deflections of Trusses by the Virtual Work Method

To develop the expression of the virtual work method that can be used to determine the deflections of trusses, consider an arbitrary statically determinate truss, as shown in Fig. 7.4(a). Let us assume that we want to determine the vertical deflection, Δ, at joint B of the truss due to the given external loads P_1 and P_2. The truss is statically determinate, so the axial forces in its members can be determined from the method of joints described previously in Chapter 4. If F represents the axial force in an arbitrary member j (e.g., member CD in Fig. 7.4(a)) of the truss, then



(a) Real System

(b) Virtual System

FIG. 7.4

(from *mechanics of materials*) the axial deformation, δ, of this member is given by

$$\delta = \frac{FL}{AE} \tag{7.19}$$

in which L, A, and E denote, respectively, the length, cross-sectional area, and modulus of elasticity of member j.

To determine the vertical deflection, Δ, at joint B of the truss, we select a virtual system consisting of a unit load acting at the joint and in the direction of the desired deflection, as shown in Fig. 7.4(b). Note that the (downward) sense of the unit load in Fig. 7.4(b) is the same as the assumed sense of the desired deflection Δ in Fig. 7.4(a). The forces in the truss members due to the virtual unit load can be determined from the method of joints. Let F_v denote the virtual force in member j. Next, we subject the truss with the virtual unit load acting on it (Fig. 7.4(b)) to the deformations of the real loads (Fig. 7.4(a)). The virtual external work performed by the virtual unit load as it goes through the real deflection Δ is equal to

$$W_{ve} = 1(\Delta) \tag{7.20}$$

To determine the virtual internal work, let us focus our attention on member j (member CD in Fig. 7.4). The virtual internal work done on member j by the virtual axial force F_v, acting through the real axial deformation δ, is equal to $F_v\delta$. Therefore, the total virtual internal work done on all the members of the truss can be written as

$$W_{vi} = \sum F_v(\delta) \tag{7.21}$$

By equating the virtual external work (Eq. (7.20)) to the virtual internal work (Eq. (7.21)) in accordance with the principle of virtual forces for deformable bodies, we obtain the following expression for the method of virtual work for truss deflections:

$$1(\Delta) = \sum F_v(\delta) \tag{7.22}$$

When the deformations are caused by external loads, Eq. (7.19) can be substituted into Eq. (7.22) to obtain

$$1(\Delta) = \sum F_v\left(\frac{FL}{AE}\right) \tag{7.23}$$

Because the desired deflection, Δ, is the only unknown in Eq. (7.23), its value can be determined by solving this equation.

Temperature Changes and Fabrication Errors

The expression of the virtual work method as given by Eq. (7.22) is quite general in the sense that it can be used to determine truss deflections due to temperature changes, fabrication errors, and any other effect for which the member axial deformations, δ, are either known or can be evaluated beforehand.

The axial deformation of a truss member j of length L due to a change in temperature (ΔT) is given by

$$\delta = \alpha(\Delta T)L \tag{7.24}$$

in which α denotes the coefficient of thermal expansion of member j. Substituting Eq. (7.24) into Eq. (7.22), we obtain the following expression:

$$1(\Delta) = \sum F_v \alpha(\Delta T)L \tag{7.25}$$

which can be used to compute truss deflections due to the changes in temperature.

Truss deflections due to fabrication errors can be determined by simply substituting changes in member lengths due to fabrication errors for δ in Eq. (7.22).

Procedure for Analysis

The following step-by-step procedure can be used to determine the deflections of trusses by the virtual work method.

1. ***Real System*** If the deflection of the truss to be determined is caused by external loads, then apply the method of joints and/or the method of sections to compute the (real) axial forces (F) in all the members of the truss. In the examples given at the end of this section, tensile member forces are considered to be positive and vice versa. Similarly, increases in temperature and increases in member lengths due to fabrication errors are considered to be positive and vice versa.
2. ***Virtual System*** Remove all the given (real) loads from the truss; then apply a unit load at the joint where the deflection is desired and in the direction of the desired deflection to form the virtual force system. By using the method of joints and/or the method of sections, compute the virtual axial forces (F_v) in all the members of the truss. The sign convention used for the virtual forces must be the same as that adopted for the real forces in step 1; that is, if real tensile forces, temperature increases, or member elongations due to fabrication errors were considered as positive in step 1, then the virtual tensile forces must also be considered to be positive and vice versa.
3. The desired deflection of the truss can now be determined by applying Eq. (7.23) if the deflection is due to external loads, Eq. (7.25) if the deflection is caused by temperature changes, or Eq. (7.22) in the case of the deflection due to fabrication errors. The application of these virtual work expressions can be facilitated by arranging the real and virtual quantities, computed in steps 1 and 2, in a tabular form, as illustrated in the following examples. A positive answer for the desired deflection means that the deflection occurs in the same direction as the unit load, whereas a negative answer indicates that the deflection occurs in the direction opposite to that of the unit load.

Procedure for Analysis

The following step-by-step procedure can be used to determine the deflections of trusses by the virtual work method.

1. ***Real System*** If the deflection of the truss to be determined is caused by external loads, then apply the method of joints and/or the method of sections to compute the (real) axial forces (F) in all the members of the truss. In the examples given at the end of this section, tensile member forces are considered to be positive and vice versa. Similarly, increases in temperature and increases in member lengths due to fabrication errors are considered to be positive and vice versa.
2. ***Virtual System*** Remove all the given (real) loads from the truss; then apply a unit load at the joint where the deflection is desired and in the direction of the desired deflection to form the virtual force system. By using the method of joints and/or the method of sections, compute the virtual axial forces (F_v) in all the members of the truss. The sign convention used for the virtual forces must be the same as that adopted for the real forces in step 1; that is, if real tensile forces, temperature increases, or member elongations due to fabrication errors were considered as positive in step 1, then the virtual tensile forces must also be considered to be positive and vice versa.
3. The desired deflection of the truss can now be determined by applying Eq. (7.23) if the deflection is due to external loads, Eq. (7.25) if the deflection is caused by temperature changes, or Eq. (7.22) in the case of the deflection due to fabrication errors. The application of these virtual work expressions can be facilitated by arranging the real and virtual quantities, computed in steps 1 and 2, in a tabular form, as illustrated in the following examples. A positive answer for the desired deflection means that the deflection occurs in the same direction as the unit load, whereas a negative answer indicates that the deflection occurs in the direction opposite to that of the unit load.

Example 7.1

Determine the horizontal deflection at joint C of the truss shown in Fig. 7.5(a) by the virtual work method.

Solution

Real System. The real system consists of the loading given in the problem, as shown in Fig. 7.5(b). The member axial forces due to the real loads (F) obtained by using the method of joints are also depicted in Fig. 7.5(b).

Virtual System. The virtual system consists of a unit (1-k) load applied in the horizontal direction at joint C, as shown in Fig. 7.5(c). The member axial forces due to the 1-k virtual load (F_v) are determined by applying the method of joints. These member forces are also shown in Fig. 7.5(c).

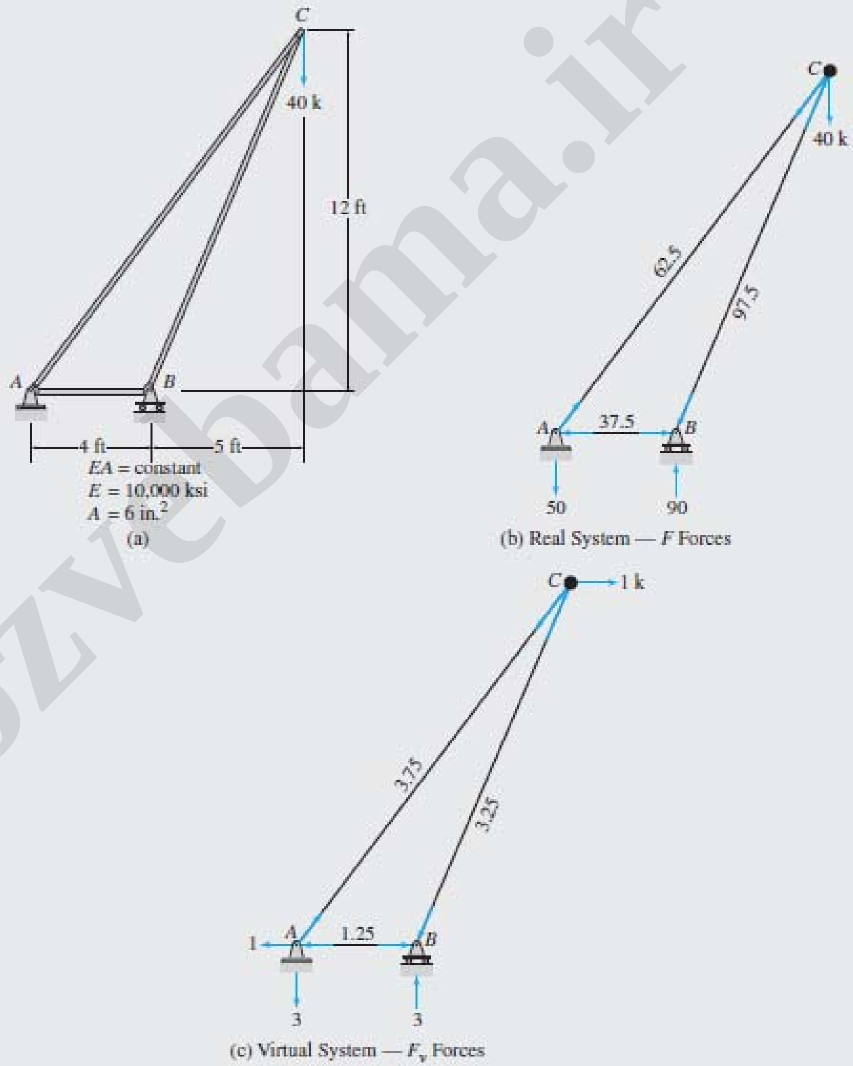


FIG. 7.5

continued

Horizontal Deflection at C, Δ_C. To facilitate the computation of the desired deflection, the real and virtual member forces are tabulated along with the member lengths (L), as shown in Table 7.1. As the values of the cross-sectional area, A, and modulus of elasticity, E, are the same for all the members, these are not included in the table. Note that the same sign convention is used for both real and virtual systems; that is, in both the third and the fourth columns of the table, tensile forces are entered as positive numbers and compressive forces as negative numbers. Then, for each member, the quantity $F_v(FL)$ is computed, and its value is entered in the fifth column of the table.

The algebraic sum of all of the entries in the fifth column, $\sum F_v(FL)$, is then determined, and its value is recorded at the bottom of the fifth column, as shown. The total virtual internal work done on all of the members of the truss is given by

$$W_{vi} = \frac{1}{EA}\sum F_v(FL)$$

The virtual external work done by the 1-k load acting through the desired horizontal deflection at C, Δ_C, is

$$W_{ve} = (1\text{ k})\Delta_C$$

Finally, we determine the desired deflection Δ_C by equating the virtual external work to the virtual internal work and solving the resulting equation for Δ_C as shown in Table 7.1. Note that the positive answer for Δ_C indicates that joint C deflects to the right, in the direction of the unit load.

TABLE 7.1

Member	L (in.)	F (k)	F_v (k)	$F_v(FL)$ ($\text{k}^2 \cdot \text{in.}$)
AB	48	−37.5	−1.25	2,250
AC	180	62.5	3.75	42,187.5
BC	156	−97.5	−3.25	49,432.5
				$\sum F_v(FL) = 93{,}870$

$$1(\Delta_C) = \frac{1}{EA}\sum F_v(FL)$$

$$(1\text{ k})\Delta_C = \frac{93{,}870\text{ k}^2\cdot\text{in.}}{(10{,}000\text{ k/in.}^2)(6\text{ in.}^2)}$$

$$\Delta_C = 1.56\text{ in.}$$

$$\Delta_C = 1.56\text{ in.} \rightarrow$$ **Ans.**

Example 7.2

Determine the horizontal deflection at joint G of the truss shown in Fig. 7.6(a) by the virtual work method.

Solution

Real System. The real system consists of the loading given in the problem, as shown in Fig. 7.6(b). The member axial forces due to the real loads (F) obtained by using the method of joints are also shown in Fig. 7.6(b).

Virtual System. The virtual system consists of a unit (1-k) load applied in the horizontal direction at joint G, as shown in Fig. 7.6(c). The member axial forces due to the 1-k virtual load (F_v) are also depicted in Fig. 7.6(c).

continued

Horizontal Deflection at C, Δ_C. To facilitate the computation of the desired deflection, the real and virtual member forces are tabulated along with the member lengths (L), as shown in Table 7.1. As the values of the cross-sectional area, A, and modulus of elasticity, E, are the same for all the members, these are not included in the table. Note that the same sign convention is used for both real and virtual systems; that is, in both the third and the fourth columns of the table, tensile forces are entered as positive numbers and compressive forces as negative numbers. Then, for each member, the quantity $F_v(FL)$ is computed, and its value is entered in the fifth column of the table.

The algebraic sum of all of the entries in the fifth column, $\sum F_v(FL)$, is then determined, and its value is recorded at the bottom of the fifth column, as shown. The total virtual internal work done on all of the members of the truss is given by

$$W_{vi} = \frac{1}{EA}\sum F_v(FL)$$

The virtual external work done by the 1-k load acting through the desired horizontal deflection at C, Δ_C, is

$$W_{ve} = (1\ \text{k})\Delta_C$$

Finally, we determine the desired deflection Δ_C by equating the virtual external work to the virtual internal work and solving the resulting equation for Δ_C as shown in Table 7.1. Note that the positive answer for Δ_C indicates that joint C deflects to the right, in the direction of the unit load.

TABLE 7.1

Member	L (in.)	F (k)	F_v (k)	$F_v(FL)$ ($\text{k}^2 \cdot \text{in.}$)
AB	48	−37.5	−1.25	2,250
AC	180	62.5	3.75	42,187.5
BC	156	−97.5	−3.25	49,432.5
				$\sum F_v(FL) = 93{,}870$

$$1(\Delta_C) = \frac{1}{EA}\sum F_v(FL)$$

$$(1\ \text{k})\Delta_C = \frac{93{,}870\ \text{k}^2 \cdot \text{in.}}{(10{,}000\ \text{k/in.}^2)(6\ \text{in.}^2)}$$

$$\Delta_C = 1.56\ \text{in.}$$

$$\Delta_C = 1.56\ \text{in.} \rightarrow$$

Ans.

Example 7.2

Determine the horizontal deflection at joint G of the truss shown in Fig. 7.6(a) by the virtual work method.

Solution

Real System. The real system consists of the loading given in the problem, as shown in Fig. 7.6(b). The member axial forces due to the real loads (F) obtained by using the method of joints are also shown in Fig. 7.6(b).

Virtual System. The virtual system consists of a unit (1-k) load applied in the horizontal direction at joint G, as shown in Fig. 7.6(c). The member axial forces due to the 1-k virtual load (F_v) are also depicted in Fig. 7.6(c).

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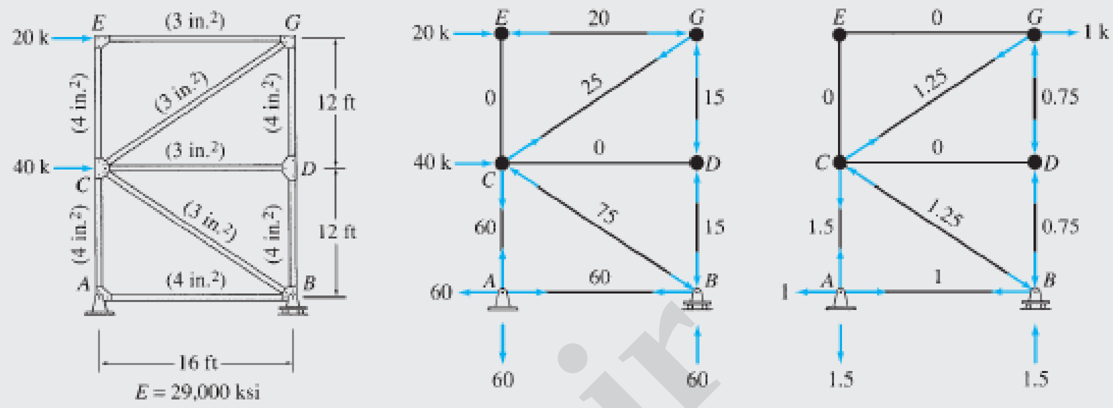


FIG. 7.6 (a) (b) Real System — F Forces (c) Virtual System — F_v Forces

Horizontal Deflection at G, Δ_G To facilitate the computation of the desired deflection, the real and virtual member forces are tabulated along with the lengths (L) and the cross-sectional areas (A) of the members, as shown in Table 7.2. The modulus of elasticity, E, is the same for all the members, so its value is not included in the table. Note that the same sign convention is used for both real and virtual systems; that is, in both the fourth and the fifth columns of the table, tensile forces are entered as positive numbers, and compressive forces as negative numbers. Then, for each member the quantity $F_v(FL/A)$ is computed, and its value is entered in the sixth column of the table. The algebraic sum of all the entries in the sixth column, $\sum F_v(FL/A)$, is then determined, and its value is recorded at the bottom of the sixth column, as shown. Finally, the desired deflection Δ_G is determined by applying the virtual work expression (Eq. (7.23)) as shown in Table 7.2. Note that the positive answer for Δ_G indicates that joint G deflects to the right, in the direction of the unit load.

TABLE 7.2

Member	L (in.)	A (in.^{2})	F (k)	F_v (k)	$F_v(FL/A)$ (k^2/in.)
AB	192	4	60	1	2,880
CD	192	3	0	0	0
EG	192	3	−20	0	0
AC	144	4	60	1.5	3,240
CE	144	4	0	0	0
BD	144	4	−15	−0.75	405
DG	144	4	−15	−0.75	405
BC	240	3	−75	−1.25	7,500
CG	240	3	25	1.25	2,500
					$\sum F_v\left(\frac{FL}{A}\right) = 16{,}930$

$$1(\Delta_G) = \frac{1}{E}\sum F_v\left(\frac{FL}{A}\right)$$

$$(1\text{ k})\Delta_G = \frac{16{,}930\text{ k}^2/\text{in.}}{29{,}000\text{ k/in.}^2}$$

$$\Delta_G = 0.584\text{ in.}$$

$$\Delta_G = 0.584\text{ in.} \rightarrow$$

Ans.

Example 7.3

Determine the horizontal and vertical components of the deflection at joint B of the truss shown in Fig. 7.7(a) by the virtual work method.

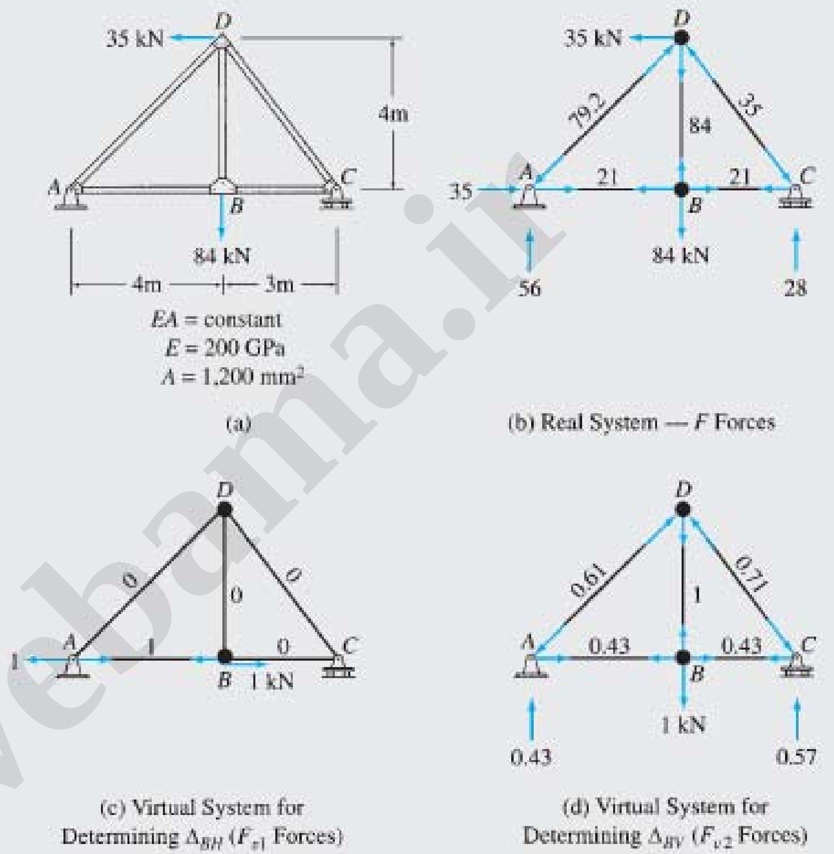


FIG. 7.7

Solution

Real System. The real system and the corresponding member axial forces (F) are shown in Fig. 7.7(b).

Horizontal Deflection at B, Δ_{BH}. The virtual system used for determining the horizontal deflection at B consists of a 1-kN load applied in the horizontal direction at joint B, as shown in Fig. 7.7(c). The member axial forces (F_{v1}) due to this virtual load are also shown in this figure. The member axial forces due to the real system (F) and this virtual system (F_{v1}) are then tabulated, and the virtual work expression given by Eq. (7.23) is applied to determine Δ_{BH}, as shown in Table 7.3.

Vertical Deflection at B, Δ_{BV}. The virtual system used for determining the vertical deflection at B consists of a 1-kN load applied in the vertical direction at joint B, as shown in Fig. 7.7(d). The member axial forces (F_{v2}) due to this virtual load are also shown in this figure. These member forces are tabulated in the sixth column of Table 7.3, and Δ_{BV} is computed by applying the virtual work expression (Eq. (7.23)), as shown in the table.

continued

TABLE 7.3

Member	L (m)	F (kN)	F_{v1} (kN)	$F_{v1}(FL)$ (kN$^2\cdot$m)	F_{v2} (kN)	$F_{v2}(FL)$ (kN$^2\cdot$m)
AB	4	21	1	84	0.43	36.12
BC	3	21	0	0	0.43	27.09
AD	5.66	−79.2	0	0	−0.61	273.45
BD	4	84	0	0	1	336.00
CD	5	−35	0	0	−0.71	124.25
		$\sum F_v(FL)$		84		796.91

$$1(\Delta_{BH}) = \frac{1}{EA}\sum F_{v1}(FL)$$

$$(1\ \text{kN})\Delta_{BH} = \frac{84}{200(10^6)(0.0012)}\ \text{kN}\cdot\text{m}$$

$$\Delta_{BH} = 0.00035\ \text{m}$$

$$\Delta_{BH} = 0.35\ \text{mm} \rightarrow \qquad \textbf{Ans.}$$

$$1(\Delta_{BV}) = \frac{1}{EA}\sum F_{v2}(FL)$$

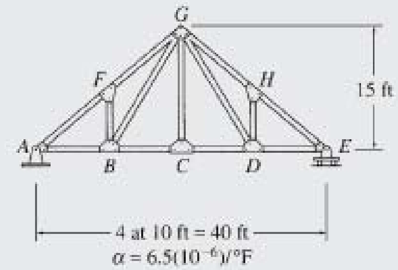
$$(1\ \text{kN})\Delta_{BV} = \frac{796.91}{200(10^6)(0.0012)}\ \text{kN}\cdot\text{m}$$

$$\Delta_{BV} = 0.00332\ \text{m}$$

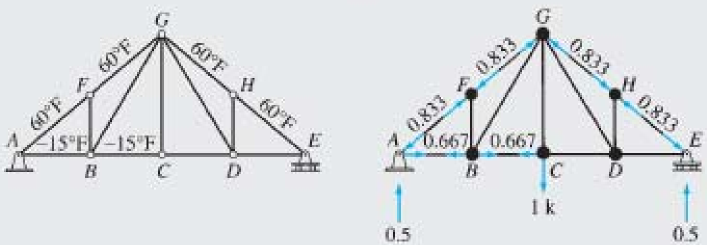
$$\Delta_{BV} = 3.32\ \text{mm} \downarrow \qquad \textbf{Ans.}$$

Example 7.4

Determine the vertical deflection at joint C of the truss shown in Fig. 7.8(a) due to a temperature drop of 15°F in members AB and BC and a temperature increase of 60°F in members AF, FG, GH, and EH. Use the virtual work method.



(a)



(b) Real System — ΔT (c) Virtual System — F_v Forces

FIG. 7.8

continued

Solution

Real System. The real system consists of the temperature changes (ΔT) given in the problem, as shown in Fig. 7.8(b).

Virtual System. The virtual system consists of a 1-k load applied in the vertical direction at joint C, as shown in Fig. 7.8(c). Note that the virtual axial forces (F_v) are computed for only those members that are subjected to temperature changes. Because the temperature changes in the remaining members of the truss are zero, their axial deformations are zero; therefore, no internal virtual work is done on those members.

Vertical Deflection at C, Δ_C. The temperature changes (ΔT) and the virtual member forces (F_v) are tabulated along with the lengths (L) of the members, in Table 7.4. The coefficient of thermal expansion, α, is the same for all the members, so its value is not included in the table. The desired deflection Δ_C is determined by applying the virtual work expression given by Eq. (7.25), as shown in the table. Note that the negative answer for Δ_C indicates that joint C deflects upward, in the direction opposite to that of the unit load.

TABLE 7.4

Member	L (ft)	ΔT (°F)	F_v (k)	$F_v(\Delta T)L$ (k-°F-ft)
AB	10	−15	0.667	−100
BC	10	−15	0.667	−100
AF	12.5	60	−0.833	−625
FG	12.5	60	−0.833	−625
GH	12.5	60	−0.833	−625
EH	12.5	60	−0.833	−625
				$\sum F_v(\Delta T)L = -2{,}700$

$$1(\Delta_C) = \alpha \sum F_v(\Delta T)L$$

$$(1\text{ k})\Delta_C = 6.5(10^{-6})(-2{,}700)\text{ k-ft}$$

$$\Delta_C = -0.0176\text{ ft} = -0.211\text{ in.}$$

$$\Delta_C = 0.211\text{ in.} \uparrow$$

Ans.

Example 7.5

Determine the vertical deflection at joint D of the truss shown in Fig. 7.9(a) if member CF is 0.6 in. too long and member EF is 0.4 in. too short. Use the method of virtual work.

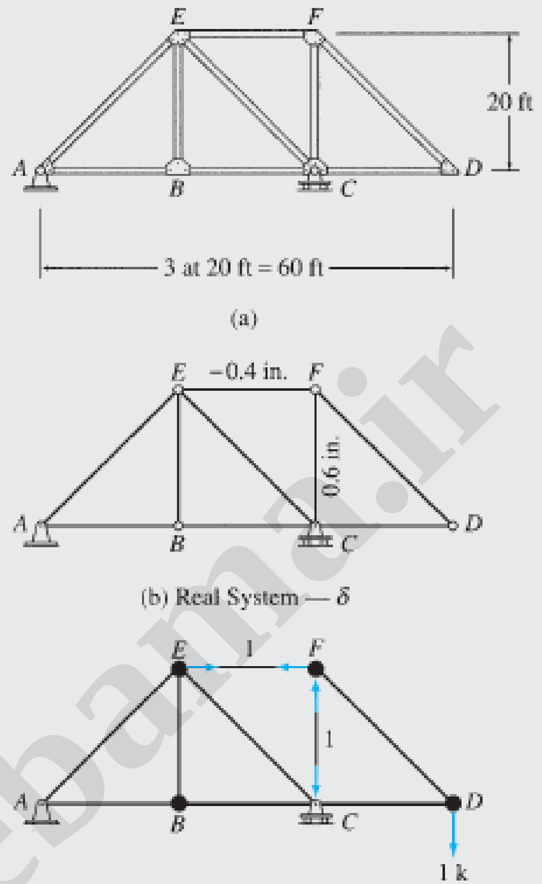
Solution

Real System. The real system consists of the changes in the lengths (δ) of members CF and EF of the truss, as shown in Fig. 7.9(b).

Virtual System. The virtual system consists of a 1-k load applied in the vertical direction at joint D, as shown in Fig. 7.9(c). The necessary virtual forces (F_v) in members CF and EF can be easily computed by using the method of sections.

Vertical Deflection at D, Δ_D. The desired deflection is determined by applying the virtual work expression given by Eq. (7.22), as shown in Table 7.5.

continued



(a)

(b) Real System — δ

(c) Virtual System — F_v Forces

FIG. 7.9

TABLE 7.5

Member	δ (in.)	F_v (k)	$F_v(\delta)$ (k-in.)
CF	0.6	−1	−0.6
EF	−0.4	1	−0.4
			$\sum F_v(\delta) = -1.0$

$$1(\Delta_D) = \sum F_v(\delta)$$

$$(1\ \text{k})\Delta_D = -1.0\ \text{k-in.}$$

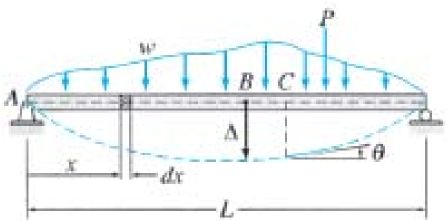
$$\Delta_D = -1.0\ \text{in.}$$

$$\Delta_D = 1.0\ \text{in.}\uparrow$$

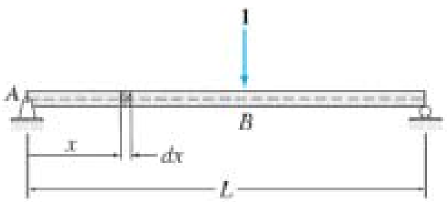
Ans.

7.4 Deflections of Beams by the Virtual Work Method

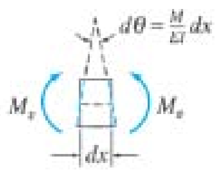
To develop an expression for the virtual work method for determining the deflections of beams, consider a beam subjected to an arbitrary loading, as shown in Fig. 7.10(a). Let us assume that the vertical deflection, Δ, at a point B of the beam is desired. To determine this deflection, we select a virtual system consisting of a unit load acting at the point and in the direction of the desired deflection, as shown in Fig. 7.10(b). Now, if we subject the beam with the virtual unit load acting on it (Fig. 7.10(b)), to the deformations due to the real loads (Fig. 7.10(a)), the virtual external work performed by the virtual unit load as it goes through the real deflection Δ is $W_{ve} = 1(\Delta)$.



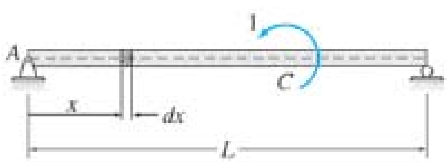
(a) Real System



(b) Virtual System for Determining Δ



(c)



(d) Virtual System for Determining θ

To obtain the virtual internal work, we focus our attention on a differential element dx of the beam located at a distance x from the left support A, as shown in Fig. 7.10(a) and (b). Because the beam with the virtual load (Fig. 7.10(b)) is subjected to the deformation due to the real loading (Fig. 7.10(a)), the virtual internal bending moment, M_v, acting on the element dx performs virtual internal work as it undergoes the real rotation $d\theta$, as shown in Fig. 7.10(c). Thus, the virtual internal work done on the element dx is given by

$$dW_{vi} = M_v(d\theta) \tag{7.26}$$

Note that because the virtual moment M_v remains constant during the real rotation $d\theta$, Eq. (7.26) does not contain a factor of 1/2. Recall from Eq. (6.10) that the change of slope $d\theta$ over the differential length dx can be expressed as

$$d\theta = \frac{M}{EI}\,dx \tag{7.27}$$

in which M = bending moment due to the real loading causing the rotation $d\theta$. By substituting Eq. (7.27) into Eq. (7.26), we write

$$dW_{vi} = M_v\left(\frac{M}{EI}\right) dx \tag{7.28}$$

The total virtual internal work done on the entire beam can now be determined by integrating Eq. (7.28) over the length L of the beam as

$$W_{vi} = \int_0^L \frac{M_v M}{EI} dx \tag{7.29}$$

By equating the virtual external work, $W_{ve} = 1(\Delta)$, to the virtual internal work (Eq. (7.29)), we obtain the following expression for the method of virtual work for beam deflections:

$$1(\Delta) = \int_0^L \frac{M_v M}{EI} dx \tag{7.30}$$

If we want the slope θ at a point C of the beam (Fig. 7.10(a)), then we use a virtual system consisting of a unit couple acting at the point, as shown in Fig. 7.10(d). When the beam with the virtual unit couple is subjected to the deformations due to the real loading, the virtual external work performed by the virtual unit couple, as it undergoes the real rotation θ, is $W_{ve} = 1(\theta)$. The expression for the internal virtual work remains the same as given in Eq. (7.29), except that M_v now denotes the bending moment due to the virtual unit couple. By setting $W_{ve} = W_{vi}$, we obtain the following expression for the method of virtual work for beam slopes:

$$1(\theta) = \int_0^L \frac{M_v M}{EI} dx \tag{7.31}$$

In the derivation of Eq. (7.29) for virtual internal work, we have neglected the internal work performed by the virtual shear forces acting through the real shear deformations. Therefore, the expressions of the virtual work method as given by Eqs. (7.30) and (7.31) do not account for the shear deformations of beams. However, for most beams (except for very deep beams), shear deformations are so small as compared to the bending deformations that their effect can be neglected in the analysis.

Procedure for Analysis

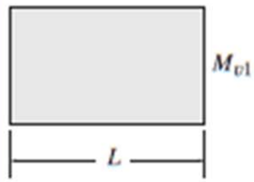
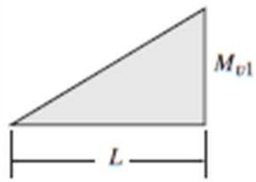
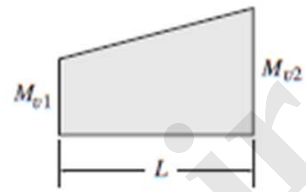
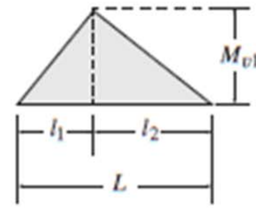
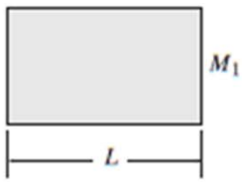
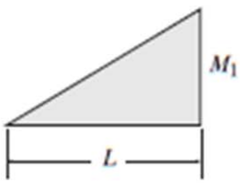
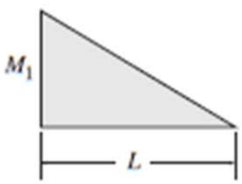
The following step-by-step procedure can be used to determine the slopes and deflections of beams by the virtual work method.

1. ***Real System*** Draw a diagram of the beam showing all the real (given) loads acting on it.
2. ***Virtual System*** Draw a diagram of the beam without the real loads. If deflection is to be determined, then apply a unit load at the point and in the direction of the desired deflection. If the slope is to be calculated, then apply a unit couple at the point on the beam where the slope is desired.
3. By examining the real and virtual systems and the variation of the flexural rigidity EI specified along the length of the beam, divide the beam into segments so that the real and virtual loadings as well as EI are continuous in each segment.
4. For each segment of the beam, determine an equation expressing the variation of the bending moment due to real loading (M) along the length of the segment in terms of a position coordinate x. The origin for x may be located anywhere on the beam and should be chosen so that the number of terms in the equation for M is minimum. It is usually convenient to consider the bending moments as positive or negative in accordance with the *beam sign convention* (Fig. 5.2).
5. For each segment of the beam, determine the equation for the bending moment due to virtual load or couple (M_v) using the same x coordinate that was used for this segment in step 4 to establish the expression for the real bending moment, M. The sign convention for the virtual bending moment (M_v) must be the same as that adopted for the real bending moment in step 4.
6. Determine the desired deflection or slope of the beam by applying the appropriate virtual work expression, Eq. (7.30) or Eq. (7.31). If the beam has been divided into segments, then the integral on the right-hand side of Eq. (7.30) or (7.31) can be evaluated by algebraically adding the integrals for all the segments of the beam.

Graphical Evaluation of Virtual Work Integrals

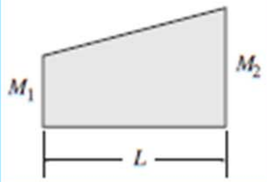
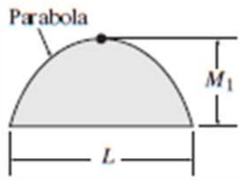
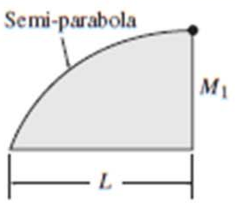
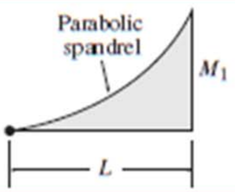
The integrals in the virtual work equations (Eqs. (7.30) and (7.31)) are generally evaluated by mathematically integrating the equations of the quantity $(M_v M / EI)$ for each segment of the structure. However, if a structure consists of segments with constant EI, and is subjected to a relatively simple loading, then an alternate graphical procedure may be more convenient for evaluating these integrals. The graphical procedure essentially involves: (a) drawing the bending moment diagrams of the structure due to the real and virtual loads; and (b) determining the expressions of the virtual work integral $(\int_0^L M_v M \, dx)$ for each segment from a table of integrals, by comparing the shapes of the segment's M and M_v diagrams with those given in the table. The expressions for such integrals for M and M_v diagrams of some simple geometric shapes are given in Table 7.6, and the graphical procedure is illustrated by Example 7.10 for beams and (in the following section) Example 7.14 for frames.

TABLE 7.6 Integrals $\int_0^L M_v M\,dx$ for Moment Diagrams of Simple Geometric Shapes

M_v / M				
	$M_{v1}M_1L$	$\frac{1}{2}M_{v1}M_1L$	$\frac{1}{2}(M_{v1}+M_{v2})M_1L$	$\frac{1}{2}M_{v1}M_1L$
	$\frac{1}{2}M_{v1}M_1L$	$\frac{1}{3}M_{v1}M_1L$	$\frac{1}{6}(M_{v1}+2M_{v2})M_1L$	$\frac{1}{6}M_{v1}M_1(L+l_1)$
	$\frac{1}{2}M_{v1}M_1L$	$\frac{1}{6}M_{v1}M_1L$	$\frac{1}{6}(2M_{v1}+M_{v2})M_1L$	$\frac{1}{6}M_{v1}M_1(L+l_2)$

continued

TABLE 7.6 (contd.)

	$\frac{1}{2}M_{o1}(M_1 + M_2)L$	$\frac{1}{6}M_{o1}(M_1 + 2M_2)L$	$\frac{1}{6}[M_{o1}(2M_1 + M_2) + M_{o2}(M_1 + 2M_2)]L$	$\frac{1}{6}M_{o1}[M_1(L + l_2) + M_2(L + l_1)]$
	$\frac{2}{3}M_{o1}M_1L$	$\frac{1}{3}M_{o1}M_1L$	$\frac{1}{3}(M_{o1} + M_{o2})M_1L$	$\frac{1}{3}M_{o1}M_1\left(L + \frac{l_1l_2}{L}\right)$
	$\frac{2}{3}M_{o1}M_1L$	$\frac{5}{12}M_{o1}M_1L$	$\frac{1}{12}(3M_{o1} + 5M_{o2})M_1L$	$\frac{1}{12}M_{o1}M_1\left(3L + 3l_1 - \frac{l_1^2}{L}\right)$
	$\frac{1}{3}M_{o1}M_1L$	$\frac{1}{4}M_{o1}M_1L$	$\frac{1}{12}(M_{o1} + 3M_{o2})M_1L$	$\frac{1}{12}M_{o1}M_1\left(L + l_1 + \frac{l_1^2}{L}\right)$

Example 7.6

Determine the slope and deflection at point A of the beam shown in Fig. 7.11(a) by the virtual work method.

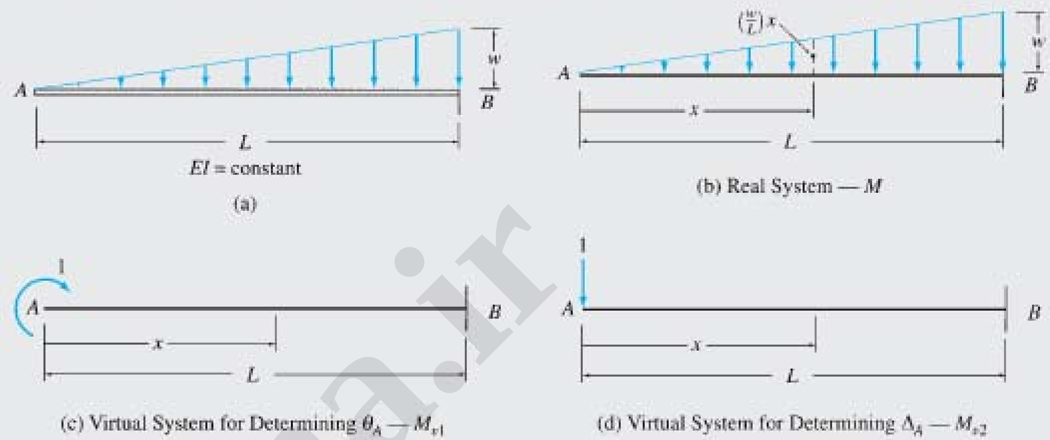


FIG. 7.11

Solution

Real System. See Fig. 7.11(b).

Slope at A, θ_A. The virtual system consists of a unit couple applied at A, as shown in Fig. 7.11(c). From Fig. 7.11(a) through (c), we can see that there are no discontinuities of the real and virtual loadings or of EI along the length of the beam. Therefore, there is no need to subdivide the beam into segments. To determine the equation for the bending moment M due to real loading, we select an x coordinate with its origin at end A of the beam, as shown in Fig. 7.11(b). By applying the method of sections described in Section 5.2, we determine the equation for M as

$$0 < x < L \qquad M = -\frac{1}{2}(x)\left(\frac{wx}{L}\right)\left(\frac{x}{3}\right) = -\frac{wx^3}{6L}$$

Similarly, the equation for the bending moment M_{v1} due to virtual unit moment in terms of the same x coordinate is

$$0 < x < L \qquad M_{v1} = 1$$

To determine the desired slope θ_A, we apply the virtual work expression given by Eq. (7.31):

$$1(\theta_A) = \int_0^L \frac{M_{v1} M}{EI} dx = \int_0^L 1\left(-\frac{wx^3}{6LEI}\right) dx$$

$$\theta_A = -\frac{w}{6EIL}\left[\frac{x^4}{4}\right]_0^L = -\frac{wL^3}{24EI}$$

The negative answer for θ_A indicates that point A rotates counterclockwise, in the direction opposite to that of the unit moment.

$$\theta_A = \frac{wL^3}{24EI} \quad \circlearrowleft \qquad \textbf{Ans.}$$

Deflection at A, Δ_A. The virtual system consists of a unit load applied at A, as shown in Fig. 7.11(d). If we use the same x coordinate as we used for computing θ_A, then the equation for M remains the same as before, and the equation for bending moment M_{v2} due to virtual unit load (Fig. 7.11(d)) is given by

$$0 < x < L \qquad M_{v2} = -1(x) = -x$$

continued

By applying the virtual work expression given by Eq. (7.30), we determine the desired deflection Δ_A as

$$1(\Delta_A) = \int_0^L \frac{M_{v2}M}{EI}dx = \int_0^L (-x)\left(-\frac{wx^3}{6LEI}\right)dx$$

$$\Delta_A = \frac{w}{6EIL}\left[\frac{x^5}{5}\right]_0^L = \frac{wL^4}{30EI}$$

The positive answer for Δ_A indicates that point A deflects downward, in the direction of the unit load.

$$\Delta_A = \frac{wL^4}{30EI}\downarrow$$ **Ans.**

Example 7.7

Determine the slope at point B of the cantilever beam shown in Fig. 7.12(a) by the virtual work method.

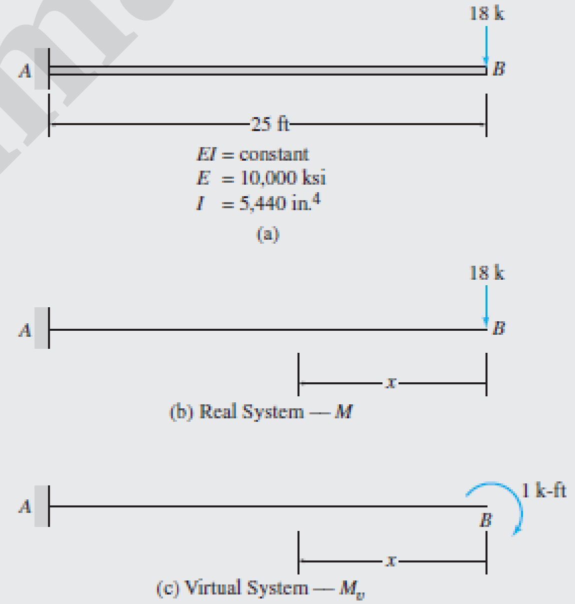


FIG. 7.12

Solution

The real and virtual systems are shown in Figs. 7.12(b) and (c), respectively. As shown in these figures, an x coordinate with its origin at end B of the beam is selected to obtain the bending moment equations. From Fig. 7.12(b), we can see that the equation for M in terms of the x coordinate is

$$0 < x < 25 \text{ ft} \qquad M = -18x$$

Similarly, from Fig. 7.12(c), we obtain the equation for M_v to be

$$0 < x < 25 \text{ ft} \qquad M_v = -1$$

continued

The slope at B can now be computed by applying the virtual work expression given by Eq. (7.31), as follows:

$$1(\theta_B) = \int_0^L \frac{M_\theta M}{EI} dx$$

$$1(\theta_B) = \frac{1}{EI}\int_0^{25} -1(-18x)\, dx$$

$$(1 \text{ k-ft})\theta_B = \frac{5{,}625 \text{ k}^2\text{-ft}^3}{EI}$$

Therefore,

$$\theta_B = \frac{5{,}625 \text{ k-ft}^2}{EI} = \frac{5{,}625(12)^2}{(10{,}000)(5{,}440)} = 0.0149 \text{ rad.}$$

The positive answer for θ_B indicates that point B rotates clockwise, in the direction of the unit moment.

$$\theta_B = 0.0149 \text{ rad.}$$ **Ans.**

Example 7.8

Determine the deflection at point D of the beam shown in Fig. 7.13(a) by the virtual work method.

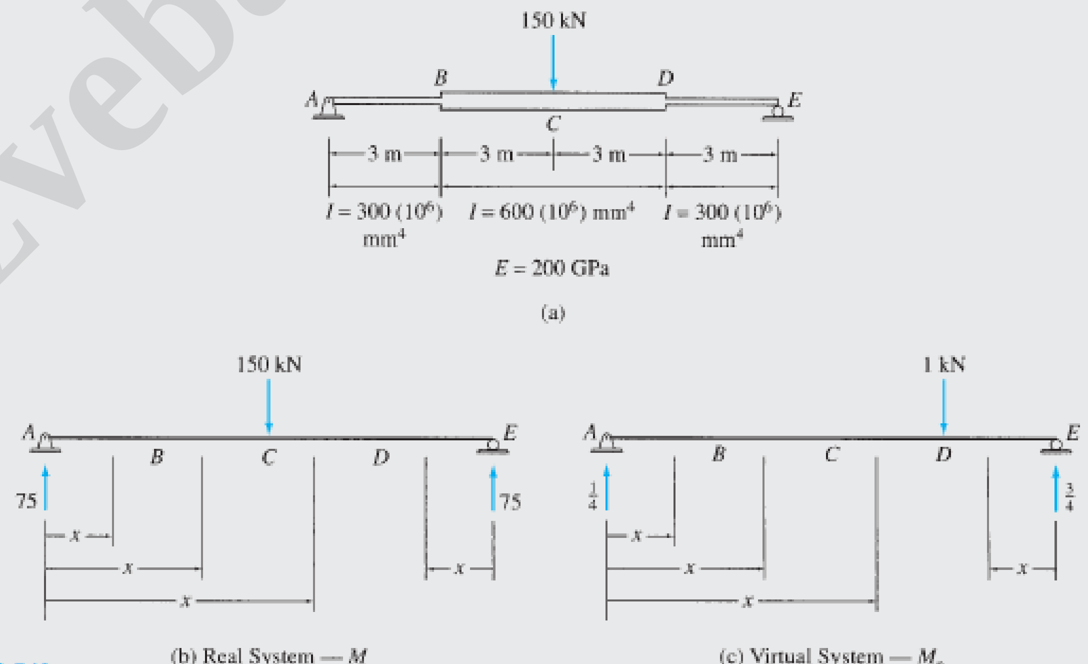


FIG. 7.13

continued

Solution

The real and virtual systems are shown in Fig. 7.13(b) and (c), respectively. It can be seen from Fig. 7.13(a) that the flexural rigidity EI of the beam changes abruptly at points B and D. Also, Fig. 7.13(b) and (c) indicates that the real and virtual loadings are discontinuous at points C and D, respectively. Consequently, the variation of the quantity $(M_v M/EI)$ will be discontinuous at points B, C, and D. Thus, the beam must be divided into four segments, AB, BC, CD, and DE; in each segment the quantity $(M_v M/EI)$ will be continuous and, therefore, can be integrated.

The x coordinates selected for determining the bending moment equations are shown in Fig. 7.13(b) and (c). Note that in any particular segment of the beam, the same x coordinate must be used to write both equations—that is, the equation for the real bending moment (M) and the equation for the virtual bending moment (M_v). The equations for M and M_v for the four segments of the beam, determined by using the method of sections, are tabulated in Table 7.7. The deflection at D can now be computed by applying the virtual work expression given by Eq. (7.30).

$$1(\Delta_D) = \int_0^L \frac{M_v M}{EI}\,dx$$

$$1(\Delta_D) = \frac{1}{EI}\left[\int_0^3 \left(\frac{x}{4}\right)(75x)\,dx + \frac{1}{2}\int_3^6 \left(\frac{x}{4}\right)(75x)\,dx + \frac{1}{2}\int_6^9 \left(\frac{x}{4}\right)(-75x+900)\,dx + \int_0^3 \left(\frac{3}{4}x\right)(75x)\,dx\right]$$

$$(1\text{ kN})\Delta_D = \frac{2{,}193.75\text{ kN}^2\cdot\text{m}^3}{EI}$$

Therefore,

$$\Delta_D = \frac{2{,}193.75\text{ kN}\cdot\text{m}^3}{EI} = \frac{2{,}193.75}{200(300)} = 0.0366\text{ m} = 36.6\text{ mm}$$

$$\Delta_D = 36.6\text{ mm}\downarrow$$ **Ans.**

TABLE 7.7

Segment	x Coordinate: Origin	x Coordinate: Limits (m)	EI ($I = 300 \times 10^6$ mm^4)	M (kN · m)	M_v (kN · m)
AB	A	0–3	EI	$75x$	$\frac{x}{4}$
BC	A	3–6	$2EI$	$75x$	$\frac{x}{4}$
CD	A	6–9	$2EI$	$75x - 150(x-6)$	$\frac{x}{4}$
ED	E	0–3	EI	$75x$	$\frac{3}{4}x$

Example 7.9

Determine the deflection at point C of the beam shown in Fig. 7.14(a) by the virtual work method.

Solution

This beam was previously analyzed by the moment-area and the conjugate-beam methods in Examples 6.7 and 6.13, respectively.

continued

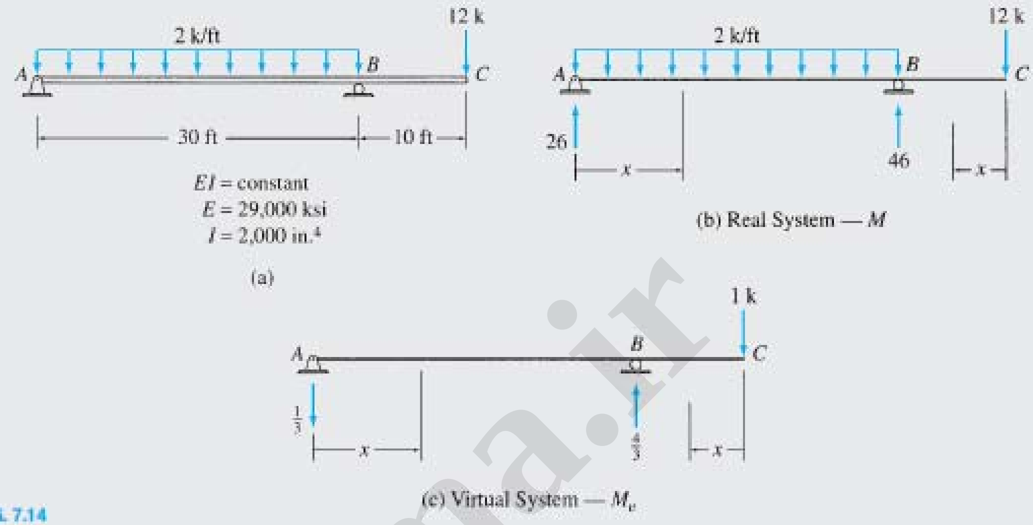


FIG. 7.14

The real and virtual systems for this problem are shown in Fig. 7.14(b) and (c), respectively. The real and virtual loadings are discontinuous at point B, so the beam is divided into two segments, AB and BC. The x coordinates used for determining the bending moment equations are shown in Fig. 7.14(b) and (c), and the equations for M and M_v obtained for each of the two segments of the beam are tabulated in Table 7.8. The deflection at C can now be determined by applying the virtual work expression given by Eq. (7.30), as follows:

TABLE 7.8

	x Coordinate			
Segment	Origin	Limits (ft)	M (k-ft)	M_v (k-ft)
AB	A	0–30	$26x - x^2$	$-\dfrac{x}{3}$
CB	C	0–10	$-12x$	$-x$

$$1(\Delta_C) = \int_0^L \frac{M_v M}{EI} dx$$

$$1(\Delta_C) = \frac{1}{EI}\left[\int_0^{30}\left(-\frac{x}{3}\right)(26x - x^2)\,dx + \int_0^{10}(-x)(-12x)\,dx\right]$$

$$(1\text{ k})\Delta_C = -\frac{6{,}500\text{ k}^2\text{-ft}^3}{EI}$$

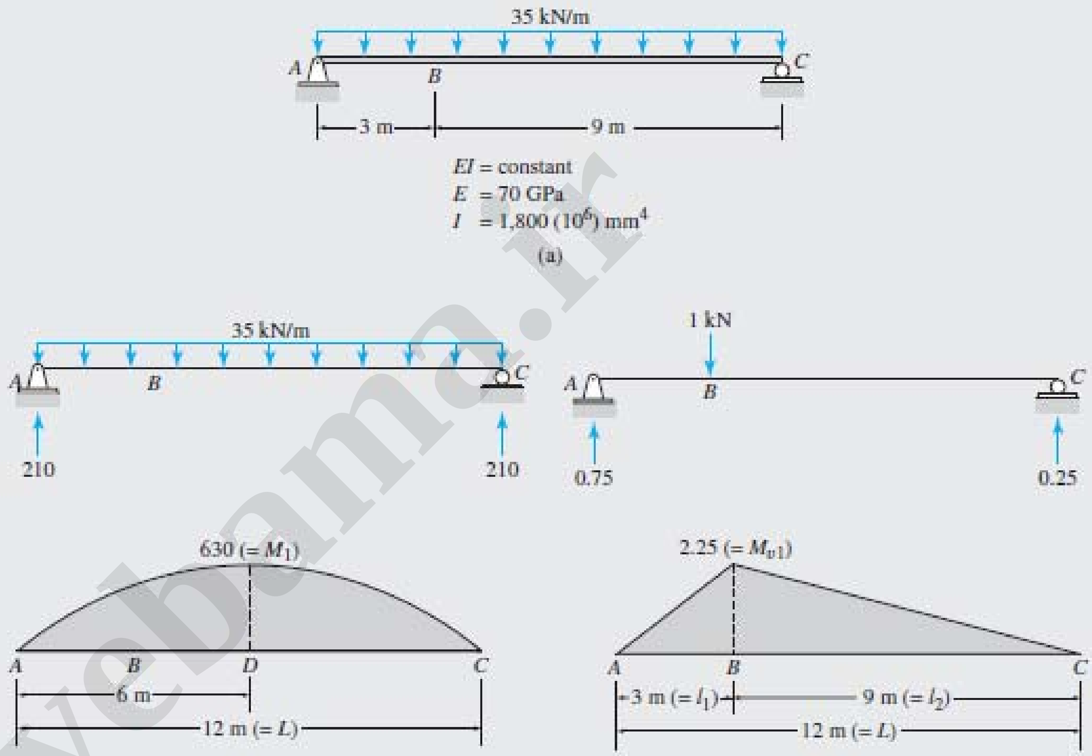
Therefore,

$$\Delta_C = -\frac{6{,}500\text{ k-ft}^3}{EI} = -\frac{6{,}500(12)^3}{(29{,}000)(2{,}000)} = -0.194\text{ in.}$$

$$\Delta_C = 0.194\text{ in.}\uparrow$$ **Ans.**

Example 7.10

Determine the deflection at point B of the beam shown in Fig. 7.15(a) by the virtual work method. Use the graphical procedure (Table 7.6) to evaluate the virtual work integral.



(b) Real System and M Diagram (kN · m)

(c) Virtual System and M_v Diagram (kN · m)

FIG. 7.15

Solution

The real and virtual systems, along with their bending moment diagrams (M and M_v), are shown in Figs. 7.15(b) and (c), respectively. As the flexural rigidity EI is constant along the length of the beam, there is no need to subdivide the beam into segments, and the virtual work equation (Eq. 7.30)) for the deflection at B can be expressed as

$$1(\Delta_B) = \frac{1}{EI}\int_0^L M_v M \, dx \qquad (1)$$

To evaluate the integral $\int_0^L M_v M \, dx$ graphically, we first compare the shape of the M diagram in Fig. 7.15(b) with the shapes listed in the left column of Table 7.6. Note that the shape of the M diagram matches the shape located in the sixth row of the table. Next, we compare the shape of the M_v diagram (Fig. 7.15(c)) with those given in the top row of the table, and notice that it is similar to the shape in the fifth column. This indicates that the expression for evaluating the integral $\int_0^L M_v M \, dx$, in this case, is located at the intersection of the sixth row and the fifth column of Table 7.6, that is,

$$\int_0^L M_v M \, dx = \frac{1}{3} M_{v1} M_1 \left(L + \frac{l_1 l_2}{L} \right)$$

continued

By substituting the numerical values of $M_{s1} = 2.25$ kN · m, $M_1 = 630$ kN · m, $L = 12$ m, $l_1 = 3$ m and $l_2 = 9$ m, into the foregoing equation, we compute the integral to be

$$\int_0^{12} M_s M\, dx = \frac{1}{3}(2.25)(630)\left(12 + \frac{3(9)}{12}\right) = 6{,}733.13 \text{ kN}^2 \cdot \text{m}^3$$

The desired deflection at B can now be conveniently determined by applying the virtual work equation (Eq. 1) as

$$(1 \text{ kN})\Delta_B = \frac{1}{EI}\int_0^{12} M_s M\, dx = \frac{6{,}733.13 \text{ kN}^2 \cdot \text{m}^3}{EI}$$

Therefore,

$$\Delta_B = \frac{6{,}733.13 \text{ kN} \cdot \text{m}^3}{EI} = \frac{6{,}733.13}{70(1{,}800)} = 0.0534 \text{ m} = 53.4 \text{ mm}$$

$$\Delta_B = 53.4 \text{ mm} \downarrow$$ **Ans.**

5 Deflections of Frames by the Virtual Work Method

Application of the virtual work method to determine the slopes and deflections of frames is similar to that for beams. To determine the deflection, Δ, or rotation, θ, at a point of a frame, a virtual unit load or unit couple is applied at that point. When the virtual system is subjected to the deformations of the frame due to real loads, the virtual external work performed by the unit load or the unit couple is $W_{ve} = 1(\Delta)$, or $W_{ve} = 1(\theta)$. As portions of the frame may undergo axial deformations in addition to the bending deformations, the total virtual internal work done on the frame is equal to the sum of the internal virtual work due to bending and that due to axial deformations. As discussed in the preceding section, when the real and virtual loadings and the flexural rigidity EI are continuous over a segment of the frame, the virtual internal work due to bending for that segment can be obtained by integrating the quantity $M_v M/EI$ over the length of the segment. The virtual internal work due to bending for the entire frame can then be obtained by summing the work for the individual segments; that is,

$$W_{vib} = \sum \int \frac{M_v M}{EI} dx \tag{7.32}$$

Similarly, if the axial forces F and F_v due to the real and virtual loads, respectively, and the axial rigidity AE are constant over the length L of a segment of the frame, then, as discussed in Section 7.3, the virtual internal work for that segment due to axial deformation is equal to $F_v(FL/AE)$. Thus, the virtual internal work due to axial deformations for the entire frame can be expressed as

$$W_{via} = \sum F_v \left(\frac{FL}{AE}\right) \tag{7.33}$$

By adding Eqs. (7.32) and (7.33), we obtain the total internal virtual work for the frame due to both bending and axial deformations as

$$W_{vi} = \sum F_v \left(\frac{FL}{AE}\right) + \sum \int \frac{M_v M}{EI} dx \tag{7.34}$$

By equating the virtual external work to the virtual internal work, we obtain the expressions for the method of virtual work for deflections and rotations of frames, respectively, as

$$1(\Delta) = \sum F_v \left(\frac{FL}{AE}\right) + \sum \int \frac{M_v M}{EI} dx \tag{7.35}$$

and

$$1(\theta) = \sum F_v \left(\frac{FL}{AE}\right) + \sum \int \frac{M_v M}{EI} dx \tag{7.36}$$

The axial deformations in the members of frames composed of common engineering materials are generally much smaller than the bending deformations and are, therefore, usually neglected in the analysis. In this text, unless stated otherwise, we will neglect the effect of axial deformations in the analysis of frames. The virtual work expressions considering only the bending deformations of frames can be obtained by simply omitting the first term on the right-hand sides of Eqs. (7.35) and (7.36), which are thus reduced to

$$1(\Delta) = \sum \int \frac{M_v M}{EI} dx \tag{7.37}$$

and

$$1(\theta) = \sum \int \frac{M_v M}{EI} dx \tag{7.38}$$

Procedure for Analysis

The following step-by-step procedure can be used to determine the slopes and deflections of frames by the virtual work method.

1. ***Real System*** Determine the internal forces at the ends of the members of the frame due to the real loading by using the procedure described in Section 5.6.
2. ***Virtual System*** If the deflection of the frame is to be determined, then apply a unit load at the point and in the direction of the desired deflection. If the rotation is to be calculated, then apply a unit couple at the point on the frame where the rotation is desired. Determine the member end forces due to the virtual loading.

3. If necessary, divide the members of the frame into segments so that the real and virtual loads and EI are continuous in each segment.
4. For each segment of the frame, determine an equation expressing the variation of the bending moment due to real loading (M) along the length of the segment in terms of a position coordinate x.
5. For each segment of the frame, determine the equation for the bending moment due to virtual load or couple (M_v) using the same x coordinate that was used for this segment in step 4 to establish the expression for the real bending moment, M. Any convenient sign convention can be used for M and M_v. However, it is important that the sign convention be the same for both M and M_v in a particular segment.
6. If the effect of axial deformations is to be included in the analysis, then go to step 7. Otherwise, determine the desired deflection or rotation of the frame by applying the appropriate virtual work expression, Eq. (7.37) or Eq. (7.38). End the analysis at this stage.
7. If necessary, divide the members of the frame into segments so that the real and virtual axial forces and AE are constant in each segment. It is not necessary that these segments be the same as those used in step 3 for evaluating the virtual internal work due to bending. It is important, however, that the same sign convention be used for both the real axial force, F, and the virtual axial force, F_v, in a particular segment.
8. Determine the desired deflection or rotation of the frame by applying the appropriate virtual work expression, Eq. (7.35) or Eq. (7.36).

Example 7.11

Determine the rotation of joint C of the frame shown in Fig. 7.16(a) by the virtual work method.

Solution

The real and virtual systems are shown in Fig. 7.16(b) and (c), respectively. The x coordinates used for determining the bending moment equations for the three segments of the frame, AB, BC, and CD, are also shown in these figures. The equations for M and M_v obtained for the three segments are tabulated in Table 7.9. The rotation of joint C of the frame can now be determined by applying the virtual work expression given by Eq. (7.38).

$$1(\theta_C) = \sum \int \frac{M_v M}{EI} dx$$

$$= \frac{1}{EI}\int_0^{30} \left(\frac{x}{30}\right)\left(38.5x - 1.5\frac{x^2}{2}\right) dx$$

$$(1 \text{ k-ft})\theta_C = \frac{6{,}487.5 \text{ k}^2\text{-ft}^3}{EI}$$

Therefore,

$$\theta_C = \frac{6{,}487.5 \text{ k-ft}^2}{EI} = \frac{6{,}487.5(12)^2}{(29{,}000)(2{,}500)} = 0.0129 \text{ rad.}$$

$$\theta_C = 0.0129 \text{ rad.}$$

Ans.

continued

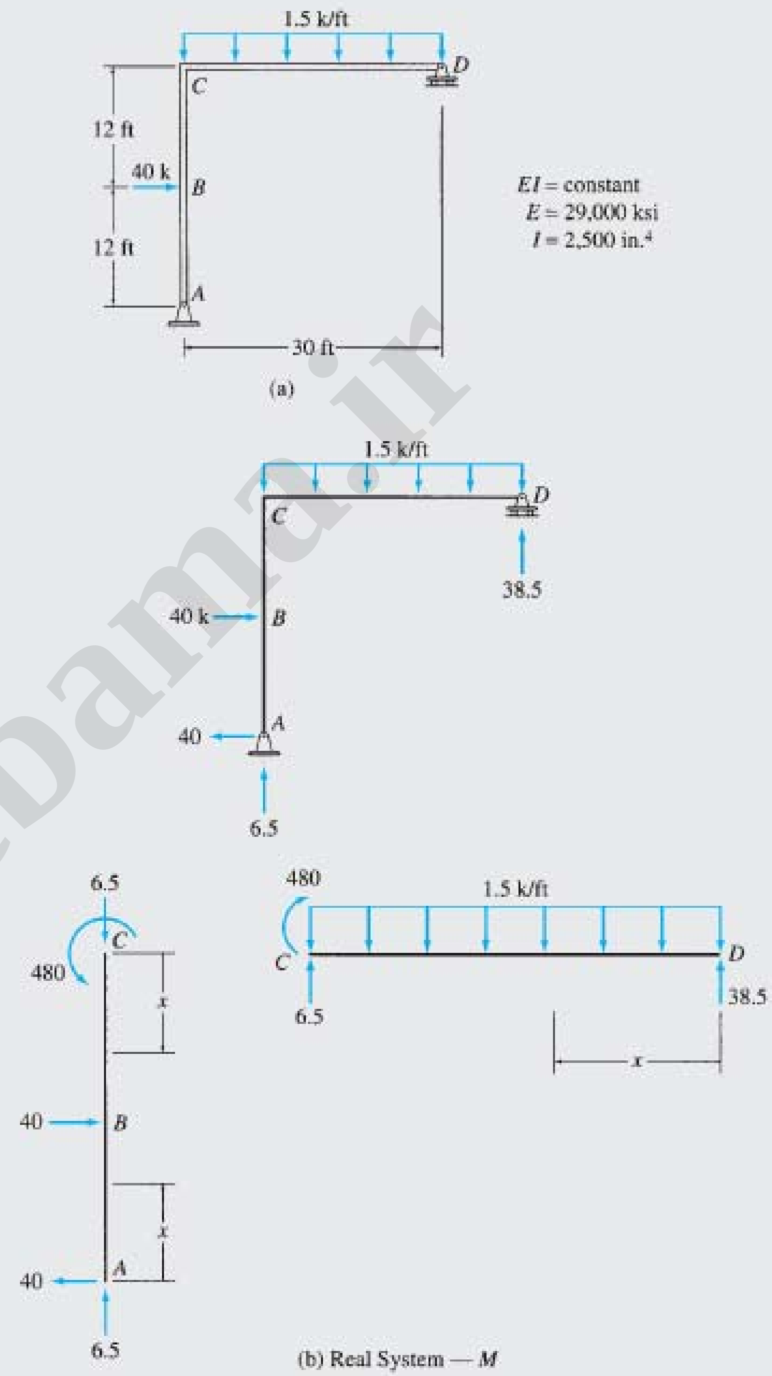


FIG. 7.16

continued

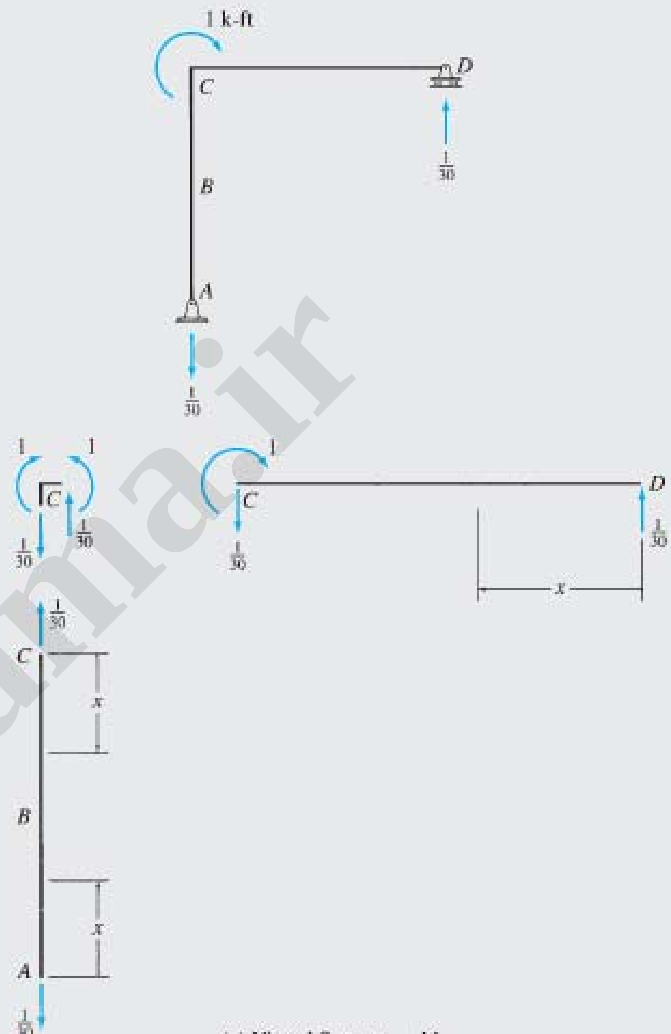


FIG. 7.16 (contd.)

TABLE 7.9

Segment	x Coordinate Origin	Limits (ft)	M (k-ft)	M_v (k-ft)
AB	A	0–12	$40x$	0
CB	C	0–12	480	0
DC	D	0–30	$38.5x - 1.5\frac{x^2}{2}$	$\frac{x}{30}$

Example 7.12

Use the virtual work method to determine the vertical deflection at joint C of the frame shown in Fig. 7.17(a).

Solution

The real and virtual systems are shown in Figs. 7.17(b) and (c), respectively. The x coordinates used for determining the bending moment equations for the two members of the frame, AB and BC, are also shown in the figures. The equations for M and M_v obtained for the two members are tabulated in Table 7.10. The vertical deflection at joint C of the frame can now be calculated by applying the virtual work expression given by Eq. (7.37):

$$1(\Delta_C) = \sum \int \frac{M_v M}{EI} dx$$

$$1(\Delta_C) = \frac{1}{EI}\left[\frac{1}{2}\int_0^5 (-4)(76x - 530)\, dx + \int_0^5 \left(-\frac{4}{5}x\right)(-6x^2)\, dx\right]$$

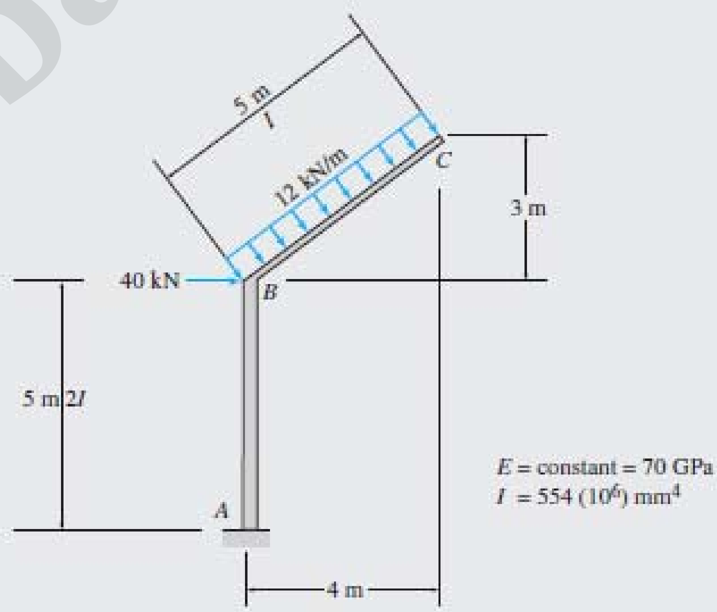
$$(1 \text{ kN})\Delta_C = \frac{4{,}150 \text{ kN}^2 \cdot \text{m}^3}{EI}$$

Therefore,

$$\Delta_C = \frac{4{,}150 \text{ kN} \cdot \text{m}^3}{EI} = \frac{4{,}150}{70(554)} = 0.107 \text{ m} = 107 \text{ mm}$$

$$\Delta_C = 107 \text{ mm} \downarrow$$

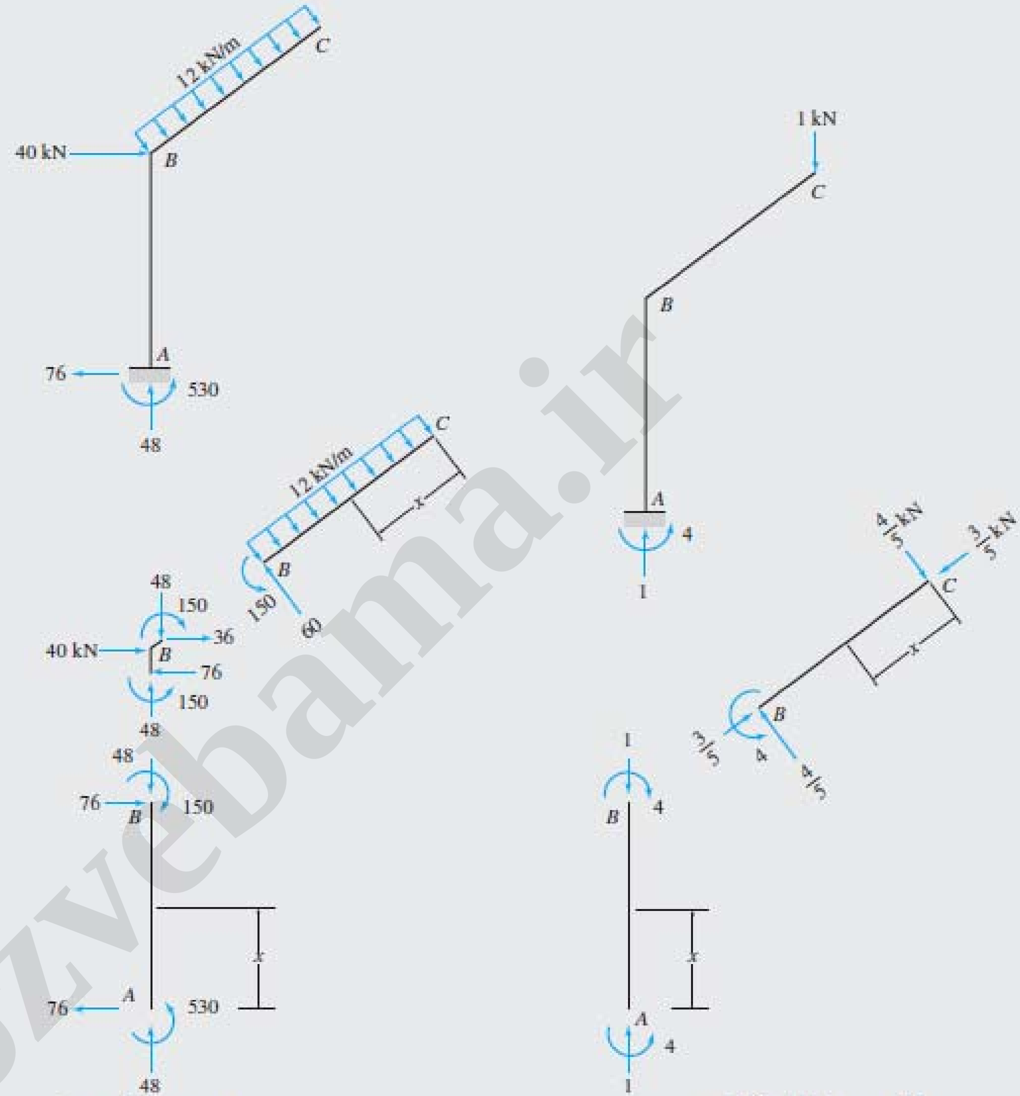
Ans.



(a)

FIG. 7.17

continued



(b) Real System — M

(c) Virtual System — M_v

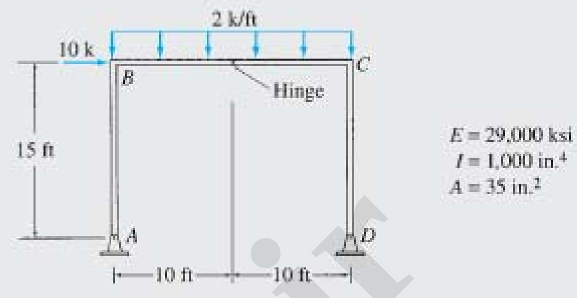
FIG. 7.17 (contd.)

TABLE 7.10

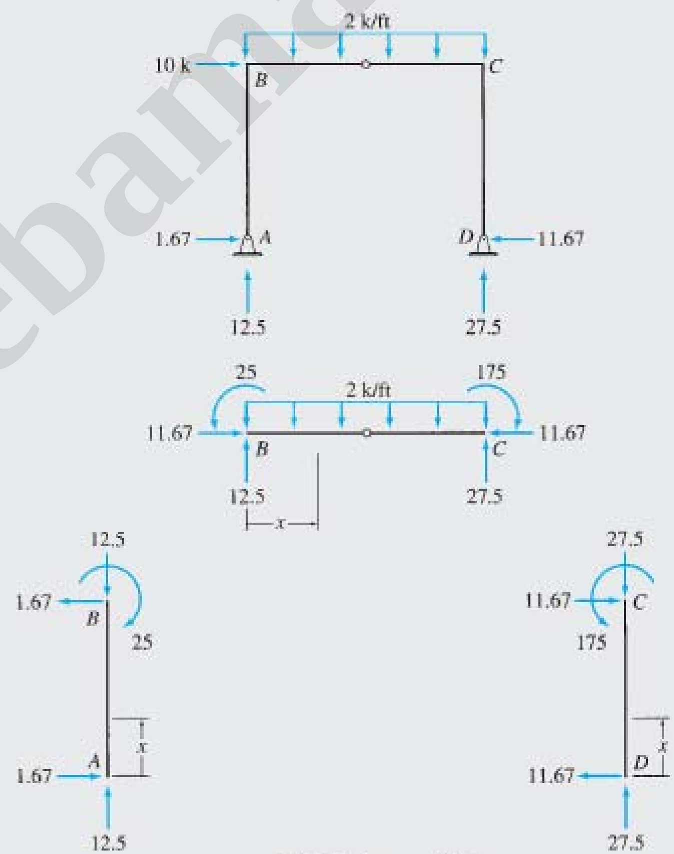
Segment	x Coordinate Origin	x Coordinate Limits (m)	EI ($I = 554 \times 10^6$ mm^4)	M (kN · m)	M_v (kN · m)
AB	A	0–5	$2EI$	$76x - 530$	-4
CB	C	0–5	EI	$-12\frac{x^2}{2}$	$-\frac{4}{5}x$

Example 7.13

Determine the horizontal deflection at joint C of the frame shown in Fig. 7.18(a) including the effect of axial deformations, by the virtual work method.



(a)



(b) Real System — M, F

FIG. 7.18

continued

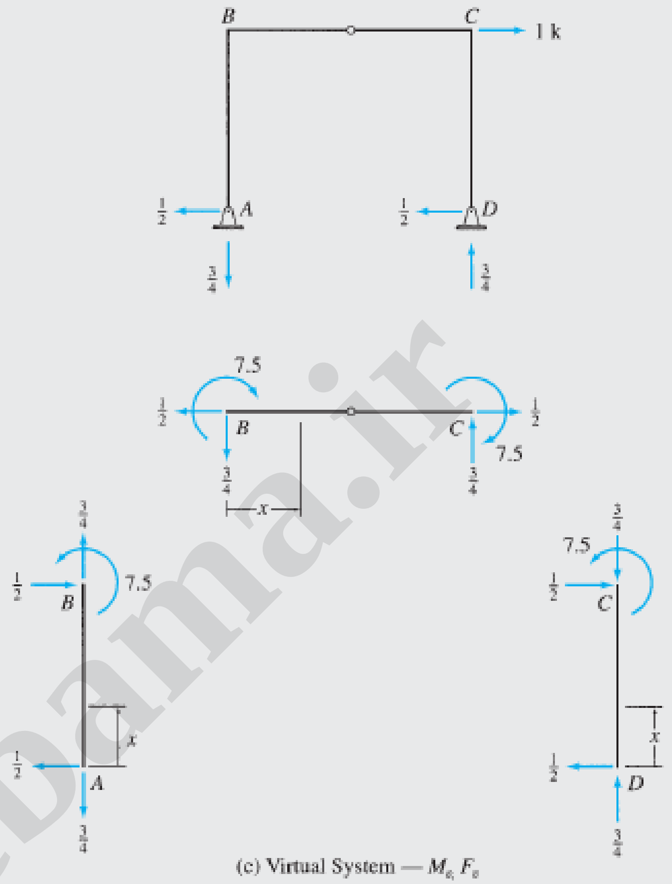


FIG. 7.18 (contd.)

TABLE 7.11

Segment	x Coordinate: Origin	x Coordinate: Limits (ft)	M (k-ft)	F (k)	M_v (k-ft)	F_v (k)
AB	A	0–15	$-1.67x$	−12.50	$\frac{x}{2}$	$\frac{3}{4}$
BC	B	0–20	$-25 + 12.5x - x^2$	−11.67	$7.5 - \frac{3}{4}x$	$\frac{1}{2}$
DC	D	0–15	$11.67x$	−27.50	$\frac{x}{2}$	$-\frac{3}{4}$

Solution

The real and virtual systems are shown in Fig. 7.18(b) and (c), respectively. The x coordinates used for determining the bending moment equations for the three members of the frame, AB, BC, and CD, are also shown in the figures. The equations for M and M_v obtained for the three members are tabulated in Table 7.11 along with the axial forces F and F_v

continued

of the members. The horizontal deflection at joint C of the frame can be determined by applying the virtual work expression given by Eq. (7.35):

$$1(\Delta_C) = \sum F_v \left(\frac{FL}{AE}\right) + \sum \int \frac{M_v M}{EI} dx$$

$$1(\Delta_C) = \frac{1}{AE}\left[\frac{3}{4}(-12.5)(15) + \frac{1}{2}(-11.67)(20) - \frac{3}{4}(-27.5)(15)\right]$$

$$+ \frac{1}{EI}\left[\int_0^{15} \frac{x}{2}(-1.67x)\,dx\right.$$

$$\left. + \int_0^{20}\left(7.5 - \frac{3}{4}x\right)(-25 + 12.5x - x^2)\,dx + \int_0^{15} \frac{x}{2}(11.67x)\,dx\right]$$

$$(1\text{ k})\Delta_C = \frac{52.08\text{ k}^2\text{-ft}}{AE} + \frac{9{,}375\text{ k}^2\text{-ft}^3}{EI}$$

Therefore,

$$\Delta_C = \frac{52.08\text{ k-ft}}{AE} + \frac{9{,}375\text{ k-ft}^3}{EI}$$

$$= \frac{52.08}{(35)(29{,}000)} + \frac{9{,}375(12)^2}{(29{,}000)(1{,}000)}$$

$$= 0.00005 + 0.04655$$

$$= 0.0466\text{ ft} = 0.559\text{ in.}$$

$$\Delta_C = 0.559\text{ in.} \rightarrow$$ **Ans.**

Note that the magnitude of the axial deformation term is negligibly small as compared to that of the bending deformation term.

Example 7.14

Determine the vertical deflection of joint A of the frame shown in Fig. 7.19(a) by the virtual work method. Use the graphical procedure (Table 7.6) to evaluate the virtual work integral.

Solution

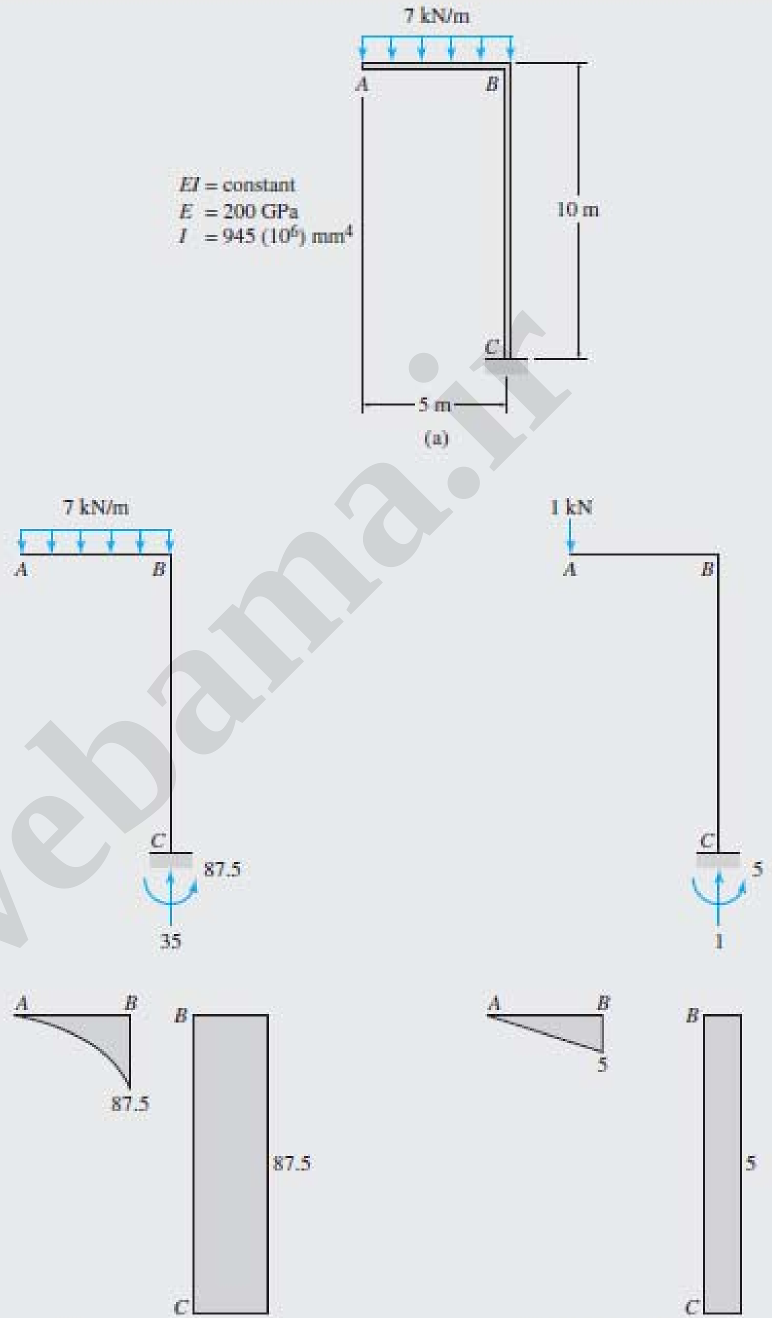
The real and virtual systems, along with their bending moment diagrams (M and M_v), are shown in Figs. 7.19(b) and (c), respectively. As the flexural rigidity EI is constant, the virtual work equation (Eq. (7.37)) can be expressed as

$$1(\Delta_A) = \frac{1}{EI}\sum \int_0^L M_v M\,dx \qquad (1)$$

To evaluate the integrals $\int_0^L M_v M\,dx$ graphically, we compare the shapes of the M and M_v diagrams for member AB with those given in Table 7.6, and obtain the relevant expression from the eighth row and second column of the table. Thus,

$$\int_0^L M_v M\,dx = \frac{1}{4} M_{v1} M_1 L$$

continued



(b) Real System and M Diagrams (kN · m)

(c) Virtual System and M_v Diagrams (kN · m)

FIG. 7.19

continued

By substituting $M_{s1} = 5$ kN·m, $M_1 = 87.5$ kN·m and $L = 5$ m, into the foregoing equation, we compute the value of the virtual work integral for member AB to be

$$\int_0^5 M_s M\,dx = \frac{1}{4}(5)(87.5)(5) = 546.9\ \text{kN}^2\cdot\text{m}^3$$

Similarly, the expression for the integral for member BC is obtained from the second row and second column of Table 7.6 as

$$\int_0^L M_s M\,dx = M_{s1} M_1 L$$

with $M_{s1} = 5$ kN·m, $M_1 = 87.5$ kN·m and $L = 10$ m, and the value of the integral for member BC is computed as

$$\int_0^{10} M_s M\,dx = (5)(87.5)(10) = 4{,}375\ \text{kN}^2\cdot\text{m}^3$$

The desired deflection at joint A can now be determined by substituting the numerical values of the integrals for the two members into the virtual work equation (Eq. 1) as

$$(1\ \text{kN})\Delta_A = \frac{1}{EI}(546.9 + 4{,}375) = \frac{4{,}921.9\ \text{kN}^2\cdot\text{m}^3}{EI}$$

Thus,

$$\Delta_A = \frac{4{,}921.9\ \text{kN}\cdot\text{m}^3}{EI} = \frac{4{,}921.9}{200(945)} = 0.026\ \text{m} = 26\ \text{mm}$$

$$\Delta_A = 26\ \text{mm}\downarrow$$ **Ans.**

7.6 Conservation of Energy and Strain Energy

Before we can develop the next method for computing deflections of structures, it is necessary to understand the concepts of conservation of energy and strain energy.

The *energy* of a structure can be simply defined as its *capacity for doing work*. The term *strain energy* is attributed to the *energy that a structure has because of its deformation*. The relationship between the work and strain energy of a structure is based on the *principle of conservation of energy*, which can be stated as follows:

> The work performed on an elastic structure in equilibrium by statically (gradually) applied external forces is equal to the work done by internal forces, or the strain energy stored in the structure.

This principle can be mathematically expressed as

$$W_e = W_i \tag{7.39}$$

or

$$W_e = U \tag{7.40}$$

In these equations, W_e and W_i represent the work done by the external and internal forces, respectively, and U denotes the strain energy of the structure. The explicit expression for the strain energy of a structure depends on the types of internal forces that can develop in the members of the structure. Such expressions for the strain energy of trusses, beams, and frames are derived in the following.

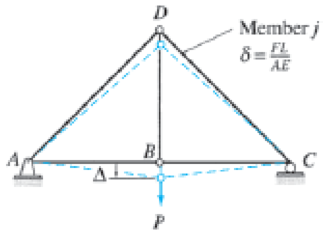


FIG. 7.20

Strain Energy of Trusses

Consider the arbitrary truss shown in Fig. 7.20. The truss is subjected to a load P, which increases gradually from zero to its final value, causing the structure to deform as shown in the figure. Because we are considering linearly elastic structures, the deflection of the truss Δ at the point of application of P increases linearly with the load; therefore, as discussed in Section 7.1 (see Fig. 7.1(c)), the external work performed by P during the deformation Δ can be expressed as

$$W_e = \frac{1}{2}P\Delta$$

To develop the expression for internal work or strain energy of the truss, let us focus our attention on an arbitrary member j (e.g., member CD in Fig. 7.20) of the truss. If F represents the axial force in this member due to the external load P, then as discussed in Section 7.3, the axial deformation of this member is given by $\delta = (FL)/(AE)$. Therefore, internal work or strain energy stored in member j, U_j, is given by

$$U_j = \frac{1}{2}F\delta = \frac{F^2L}{2AE}$$

The strain energy of the entire truss is simply equal to the sum of the strain energies of all of its members and can be expressed as

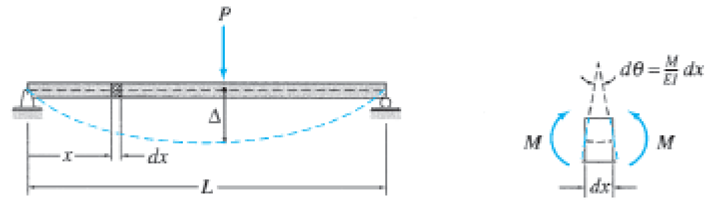
$$U = \sum \frac{F^2L}{2AE} \qquad (7.41)$$

Note that a factor of $\frac{1}{2}$ appears in the expression for strain energy because the axial force F and the axial deformation δ caused by F in each member of the truss are related by the linear relationship $\delta = (FL)/(AE)$.

Strain Energy of Beams

To develop the expression for the strain energy of beams, consider an arbitrary beam, as shown in Fig. 7.21(a). As the external load P acting on the beam increases gradually from zero to its final value, the internal bending moment M acting on a differential element dx of the beam (Fig. 7.21(a) and (b)) also increases gradually from zero to its final value, while the cross sections of element dx rotate by an angle $d\theta$ with respect to each other. The internal work or the strain energy stored in the element dx is, therefore, given by

$$dU = \frac{1}{2}M(d\theta) \qquad (7.42)$$



(a) (b)

Recalling from Section 7.4 (Eq. (7.27)) that the change in slope, $d\theta$, can be expressed in terms of the bending moment, M, by the relationship $d\theta = (M/EI)\,dx$, we write Eq. (7.42) as

$$dU = \frac{M^2}{2EI}dx \tag{7.43}$$

The expression for the strain energy of the entire beam can now be obtained by integrating Eq. (7.43) over the length L of the beam:

$$U = \int_0^L \frac{M^2}{2EI}dx \tag{7.44}$$

When the quantity M/EI is not a continuous function of x over the entire length of the beam, then the beam must be divided into segments so that M/EI is continuous in each segment. The integral on the right-hand side of Eq. (7.44) is then evaluated by summing the integrals for all the segments of the beam. We must realize that Eq. (7.44) is based on the consideration of bending deformations of beams and does not include the effect of shear deformations, which, as stated previously, are negligibly small as compared to the bending deformations for most beams.

Strain Energy of Frames

The portions of frames may be subjected to axial forces as well as bending moments, so the total strain energy (U) of frames is expressed as the sum of the strain energy due to axial forces (U_a) and the strain energy due to bending (U_b); that is,

$$U = U_a + U_b \tag{7.45}$$

If a frame is divided into segments so that the quantity F/AE is constant over the length L of each segment, then—as shown previously in the case of trusses—the strain energy stored in each segment due to the axial force F is equal to $(F^2L)/(2AE)$. Therefore, the strain energy due to axial forces for the entire frame can be expressed as

$$U_a = \sum \frac{F^2L}{2AE} \tag{7.46}$$

Similarly, if the frame is divided into segments so that the quantity M/EI is continuous over each segment, then the strain energy stored in each segment due to bending can be obtained by integrating the quantity M/EI over the length of the segment (Eq. (7.44)). The strain energy due to bending for the entire frame is equal to the sum of strain energies of bending of all the segments of the frame and can be expressed as

$$U_b = \sum \int \frac{M^2}{2EI} dx \tag{7.47}$$

By substituting Eqs. (7.46) and (7.47) into Eq. (7.45), we obtain the following expression for the strain energy of frames due to both the axial forces and bending:

$$U = \sum \frac{F^2 L}{2AE} + \sum \int \frac{M^2}{2EI} dx \tag{7.48}$$

As stated previously, the axial deformations of frames are generally much smaller than the bending deformations and are usually neglected in the analysis. The strain energy of frames due only to bending is expressed as

$$U = \sum \int \frac{M^2}{2EI} dx \tag{7.49}$$

7.7 Castigliano's Second Theorem

In this section, we consider another energy method for determining deflections of structures. This method, which can be applied only to linearly elastic structures, was initially presented by Alberto Castigliano in 1873 and is commonly known as *Castigliano's second theorem*. (Castigliano's first theorem, which can be used to establish equations of equilibrium of structures, is not considered in this text.) Castigliano's second theorem can be stated as follows:

> For linearly elastic structures, the partial derivative of the strain energy with respect to an applied force (or couple) is equal to the displacement (or rotation) of the force (or couple) along its line of action.

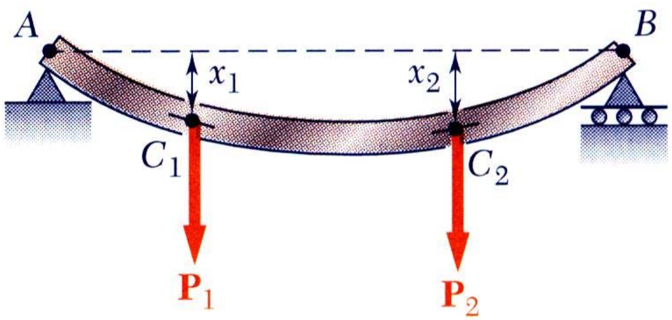
In mathematical form, this theorem can be stated as:

$$\frac{\partial U}{\partial P_i} = \Delta_i \quad \text{or} \quad \frac{\partial U}{\partial \overline{M}_i} = \theta_i \tag{7.50}$$

in which U = strain energy; Δ_i = deflection of the point of application of the force P_i in the direction of P_i; and θ_i = rotation of the point of application of the couple $\overline{M}_i$ in the direction of $\overline{M}_i$.

To prove this theorem, consider the beam shown in Fig. 7.22. The beam is subjected to external loads P_1, P_2, and P_3, which increase gradually from zero to their final values, causing the beam to deflect, as

Castigliano's Theorem



- Strain energy for any elastic structure subjected to two concentrated loads,

$$U = \frac{1}{2}\left(\alpha_{11}P_1^2 + 2\alpha_{12}P_1P_2 + \alpha_{22}P_2^2\right)$$

- Differentiating with respect to the loads,

$$\frac{\partial U}{\partial P_1} = \alpha_{11}P_1 + \alpha_{12}P_2 = x_1$$

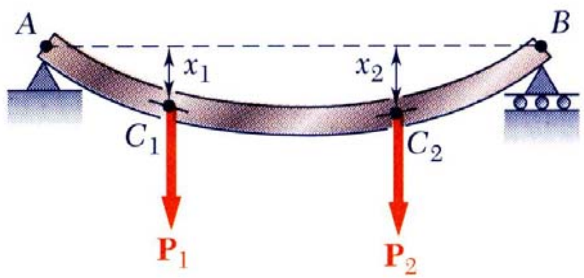
$$\frac{\partial U}{\partial P_2} = \alpha_{12}P_1 + \alpha_{22}P_2 = x_2$$

- *Castigliano's theorem*: For an elastic structure subjected to n loads, the deflection x_j of the point of application of P_j can be expressed as

$$x_j = \frac{\partial U}{\partial P_j} \quad \text{and} \quad \theta_j = \frac{\partial U}{\partial M_j} \quad \phi_j = \frac{\partial U}{\partial T_j}$$

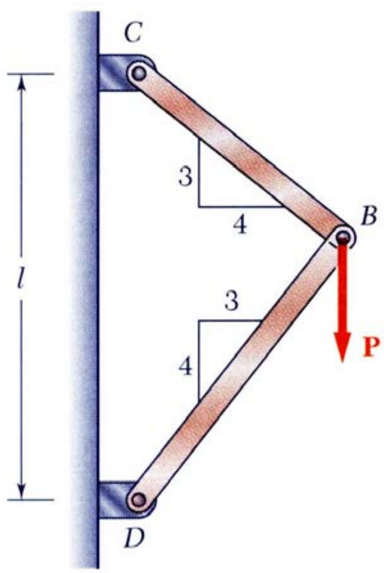
Deflections by Castigliano's Theorem

- Application of Castigliano's theorem is simplified if the differentiation with respect to the load P_j is performed before the integration or summation to obtain the strain energy U.



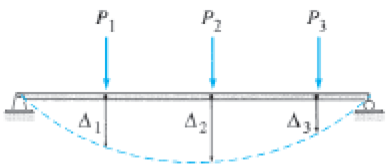
- In the case of a beam,

$$U = \int_0^L \frac{M^2}{2EI} dx \qquad x_j = \frac{\partial U}{\partial P_j} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial P_j} dx$$



- For a truss,

$$U = \sum_{i=1}^{n} \frac{F_i^2 L_i}{2 A_i E} \qquad x_j = \frac{\partial U}{\partial P_j} = \sum_{i=1}^{n} \frac{F_i L_i}{A_i E} \frac{\partial F_i}{\partial P_j}$$



shown in the figure. The strain energy (U) stored in the beam due to the external work (W_e) performed by these forces is given by

$$U = W_e = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_3\Delta_3 \tag{7.51}$$

in which Δ_1, Δ_2, and Δ_3 are the deflections of the beam at the points of application of P_1, P_2, and P_3, respectively, as shown in the figure. As Eq. (7.51) indicates, the strain energy U is a function of the external loads and can be expressed as

$$U = f(P_1, P_2, P_3) \tag{7.52}$$

Now, assume that the deflection Δ_2 of the beam at the point of application of P_2 is to be determined. If P_2 is increased by an infinitesimal amount dP_2, then the increase in strain energy of the beam due to the application of dP_2 can be written as

$$dU = \frac{\partial U}{\partial P_2} dP_2 \tag{7.53}$$

and the total strain energy, U_T, now stored in the beam is given by

$$U_T = U + dU = U + \frac{\partial U}{\partial P_2} dP_2 \tag{7.54}$$

The beam is assumed to be composed of linearly elastic material, so regardless of the sequence in which the loads $P_1, (P_2 + dP_2)$, and P_3 are applied, the total strain energy stored in the beam should be the same.

Consider, for example, the sequence in which dP_2 is applied to the beam before the application of P_1, P_2, and P_3. If $d\Delta_2$ is the deflection of the beam at the point of application of dP_2 due to dP_2, then the strain energy stored in the beam is given by $(1/2)(dP_2)(d\Delta_2)$. The loads P_1, P_2, and P_3 are then applied to the beam, causing the additional deflections Δ_1, Δ_2, and Δ_3, respectively, at their points of application. Note that since the beam is linearly elastic, the loads P_1, P_2, and P_3 cause the same deflections, Δ_1, Δ_2, and Δ_3, respectively, and perform the same amount of external work on the beam regardless of whether any other load is acting on the beam or not. The total strain energy stored in the beam during the application of dP_2 followed by P_1, P_2, and P_3 is given by

$$U_T = \frac{1}{2}(dP_2)(d\Delta_2) + dP_2(\Delta_2) + \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_3\Delta_3 \tag{7.55}$$

Since dP_2 remains constant during the additional deflection, Δ_2, of its point of application, the term $dP_2(\Delta_2)$ on the right-hand side of Eq. (7.55) does not contain the factor 1/2. The term $(1/2)(dP_2)(d\Delta_2)$ represents a small quantity of second order, so it can be neglected, and Eq. (7.55) can be written as

$$U_T = dP_2(\Delta_2) + \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_3\Delta_3 \tag{7.56}$$

By substituting Eq. (7.51) into Eq. (7.56) we obtain

$$U_T = dP_2(\Delta_2) + U \tag{7.57}$$

and by equating Eqs. (7.54) and (7.57), we write

$$U + \frac{\partial U}{\partial P_2}dP_2 = dP_2(\Delta_2) + U$$

or

$$\frac{\partial U}{\partial P_2} = \Delta_2$$

which is the mathematical statement of Castigliano's second theorem.

Application to Trusses

To develop the expression of Castigliano's second theorem, which can be used to determine the deflections of trusses, we substitute Eq. (7.41) for the strain energy (U) of trusses into the general expression of Castigliano's second theorem for deflections as given by Eq. (7.50) to obtain

$$\Delta = \frac{\partial}{\partial P}\sum\frac{F^2L}{2AE} \tag{7.58}$$

As the partial derivative $\partial F^2/\partial P = 2F(\partial F/\partial P)$, the expression of Castigliano's second theorem for trusses can be written as

$$\Delta = \sum\left(\frac{\partial F}{\partial P}\right)\frac{FL}{AE} \tag{7.59}$$

The foregoing expression is similar in form to the expression of the method of virtual work for trusses (Eq. (7.23)). As illustrated by the solved examples at the end of this section, the procedure for computing deflections by Castigliano's second theorem is also similar to that of the virtual work method.

Application to Beams

By substituting Eq. (7.44) for the strain energy (U) of beams into the general expressions of Castigliano's second theorem (Eq. (7.50)), we obtain the following expressions for the deflections and rotations, respectively, of beams:

$$\Delta = \frac{\partial}{\partial P}\int_0^L \frac{M^2}{2EI}dx \quad \text{and} \quad \theta = \frac{\partial}{\partial \bar{M}}\int_0^L \frac{M^2}{2EI}dx$$

or

$$\Delta = \int_0^L \left(\frac{\partial M}{\partial P}\right)\frac{M}{EI}dx \qquad (7.60)$$

and

$$\theta = \int_0^L \left(\frac{\partial M}{\partial \overline{M}}\right)\frac{M}{EI}dx \qquad (7.61)$$

Application to Frames

Similarly, by substituting Eq. (7.48) for the strain energy (U) of frames due to the axial forces and bending into the general expressions of Castigliano's second theorem (Eq. (7.50)), we obtain the following expressions for the deflections and rotations, respectively, of frames:

$$\Delta = \sum \left(\frac{\partial F}{\partial P}\right)\frac{FL}{AE} + \sum \int \left(\frac{\partial M}{\partial P}\right)\frac{M}{EI}dx \qquad (7.62)$$

and

$$\theta = \sum \left(\frac{\partial F}{\partial \overline{M}}\right)\frac{FL}{AE} + \sum \int \left(\frac{\partial M}{\partial \overline{M}}\right)\frac{M}{EI}dx \qquad (7.63)$$

When the effect of axial deformations of the members of frames is neglected in the analysis, Eqs. (7.62) and (7.63) reduce to

$$\Delta = \sum \int \left(\frac{\partial M}{\partial P}\right)\frac{M}{EI}dx \qquad (7.64)$$

and

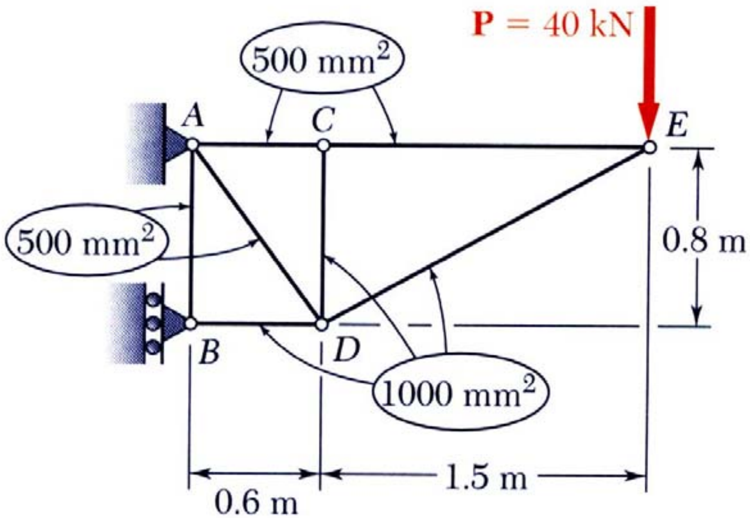
$$\theta = \sum \int \left(\frac{\partial M}{\partial \overline{M}}\right)\frac{M}{EI}dx \qquad (7.65)$$

Procedure for Analysis

As stated previously, the procedure for computing deflections of structures by Castigliano's second theorem is similar to that of the virtual work method. The procedure essentially involves the following steps.

1. If an external load (or couple) is acting on the given structure at the point and in the direction of the desired deflection (or rotation), then designate that load (or couple) as the variable P (or $\bar{M}$) and go to step 2. Otherwise, apply a fictitious load P (or couple $\bar{M}$) at the point and in the direction of the desired deflection (or rotation).
2. Determine the axial force F and/or the equation(s) for bending moment $M(x)$ in each member of the structure in terms of P (or $\bar{M}$).
3. Differentiate the member axial forces F and/or the bending moments $M(x)$ obtained in step 2 with respect to the variable P (or $\bar{M}$) to compute $\partial F/\partial P$ and/or $\partial M/\partial P$ (or $\partial F/\partial \bar{M}$ and/or $\partial M/\partial \bar{M}$).
4. Substitute the numerical value of P (or $\bar{M}$) into the expressions of F and/or $M(x)$ and their partial derivatives. If P (or $\bar{M}$) represents a fictitious load (or couple), its numerical value is zero.
5. Apply the appropriate expression of Castigliano's second theorem (Eqs. (7.59) through (7.65)) to determine the desired deflection or rotation of the structure. A positive answer for the desired deflection (or rotation) indicates that the deflection (or rotation) occurs in the same direction as P (or $\bar{M}$) and vice versa.

Sample Problem 11.5

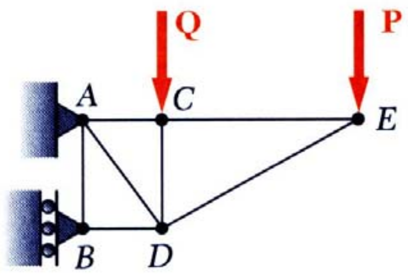


Members of the truss shown consist of sections of aluminum pipe with the cross-sectional areas indicated. Using $E = 73$ GPa, determine the vertical deflection of the joint C caused by the load P.

SOLUTION:

- For application of Castigliano's theorem, introduce a dummy vertical load Q at C. Find the reactions at A and B due to the dummy load from a free-body diagram of the entire truss.
- Apply the method of joints to determine the axial force in each member due to Q.
- Combine with the results of Sample Problem 11.4 to evaluate the derivative with respect to Q of the strain energy of the truss due to the loads P and Q.
- Setting $Q = 0$, evaluate the derivative which is equivalent to the desired displacement at C.

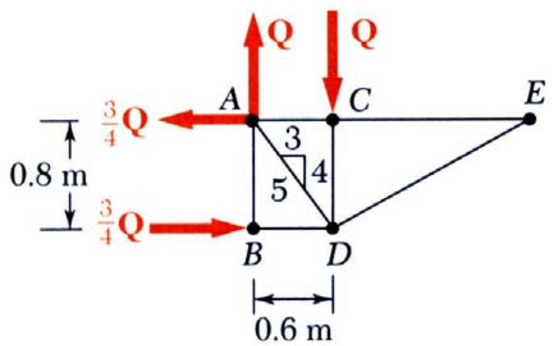
Sample Problem 11.5



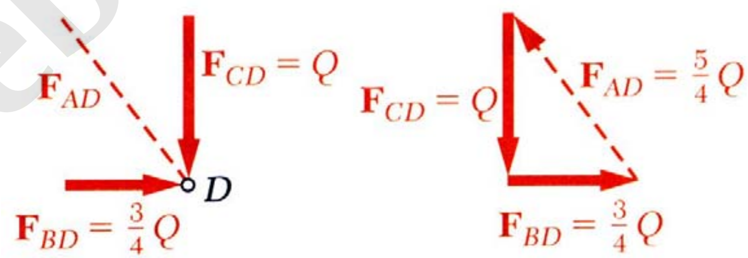
SOLUTION:

- Find the reactions at A and B due to a dummy load Q at C from a free-body diagram of the entire truss.

$$A_x = -\tfrac{3}{4}Q \qquad A_y = Q \qquad B = \tfrac{3}{4}Q$$



- Apply the method of joints to determine the axial force in each member due to Q.

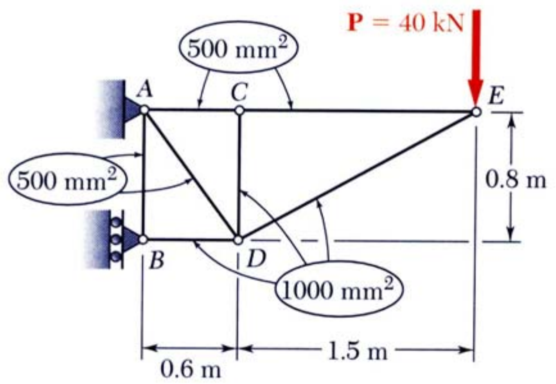


$$F_{CE} = F_{DE} = 0$$

$$F_{AC} = 0;\ F_{CD} = -Q$$

$$F_{AB} = 0;\ F_{BD} = -\tfrac{3}{4}Q$$

Sample Problem 11.5



Member	F_i	$\partial F_i/\partial Q$	L_i, m	A_i, m²	$\left(\frac{F_iL_i}{A_i}\right)\frac{\partial F_i}{\partial Q}$
AB	0	0	0.8	500×10^{-6}	0
AC	$+15P/8$	0	0.6	500×10^{-6}	0
AD	$+5P/4 + 5Q/4$	$\frac{5}{4}$	1.0	500×10^{-6}	$+3125P + 3125Q$
BD	$-21P/8 - 3Q/4$	$-\frac{3}{4}$	0.6	1000×10^{-6}	$+1181P + 338Q$
CD	$-Q$	-1	0.8	1000×10^{-6}	$+ 800Q$
CE	$+15P/8$	0	1.5	500×10^{-6}	0
DE	$-17P/8$	0	1.7	1000×10^{-6}	0

- Combine with the results of Sample Problem 11.4 to evaluate the derivative with respect to Q of the strain energy of the truss due to the loads P and Q.

$$y_C = \sum \left(\frac{F_i L_i}{A_i E}\right)\frac{\partial F_i}{\partial Q} = \frac{1}{E}(4306P + 4263Q)$$

- Setting $Q = 0$, evaluate the derivative which is equivalent to the desired displacement at C.

$$y_C = \frac{4306\left(40 \times 10^3 N\right)}{73 \times 10^9 \text{Pa}}$$

$$y_C = 2.36 \text{ mm} \downarrow$$

Example 7.15

Determine the deflection at point C of the beam shown in Fig. 7.23(a) by Castigliano's second theorem.

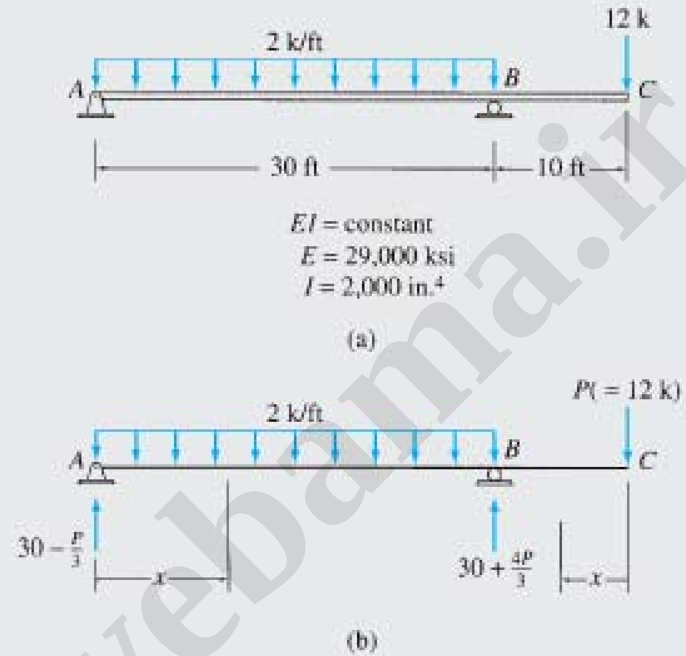


FIG. 7.23

Solution

This beam was previously analyzed by the moment-area, the conjugate-beam, and the virtual work methods in Examples 6.7, 6.13, and 7.9, respectively.

The 12-k external load is already acting at point C, where the deflection is to be determined, so we designate this load as the variable P, as shown in Fig. 7.23(b). Next, we compute the reactions of the beam in terms of P. These are also shown in Fig. 7.23(b). Since the loading is discontinuous at point B, the beam is divided into two segments, AB and BC. The x coordinates used for determining the equations for the bending moment in the two segments of the beam are shown in Fig. 7.23(b). The equations for M (in terms of P) obtained for the segments of the beam are tabulated in Table 7.12, along with the partial derivatives of M with respect to P.

continued

TABLE 7.12

Segment	x Coordinate: Origin	x Coordinate: Limits (ft)	M (k-ft)	$\frac{\partial M}{\partial P}$ (k-ft/k)
AB	A	0–30	$\left(30-\frac{P}{3}\right)x-x^2$	$-\frac{x}{3}$
CB	C	0–10	$-Px$	$-x$

The deflection at C can now be determined by substituting $P = 12$ k into the equations for M and $\partial M/\partial P$ and by applying the expression of Castigliano's second theorem as given by Eq. (7.60):

$$\Delta_C = \int_0^L \left(\frac{\partial M}{\partial P}\right)\left(\frac{M}{EI}\right) dx$$

$$\Delta_C = \frac{1}{EI}\left[\int_0^{30}\left(-\frac{x}{3}\right)\left(30x-\frac{12x}{3}-x^2\right)dx+\int_0^{10}(-x)(-12x)\,dx\right]$$

$$= \frac{1}{EI}\left[\int_0^{30}\left(-\frac{x}{3}\right)(26x-x^2)\,dx+\int_0^{10}(-x)(-12x)\,dx\right]$$

$$= -\frac{6{,}500 \text{ k-ft}^3}{EI} = -\frac{6{,}500(12)^3}{(29{,}000)(2{,}000)} = -0.194 \text{ in.}$$

The negative answer for Δ_C indicates that point C deflects upward in the direction opposite to that of P.

$$\Delta_C = 0.194 \text{ in.} \uparrow$$ **Ans.**

Example 7.16

Use Castigliano's second theorem to determine the deflection at point B of the beam shown in Fig. 7.24(a).

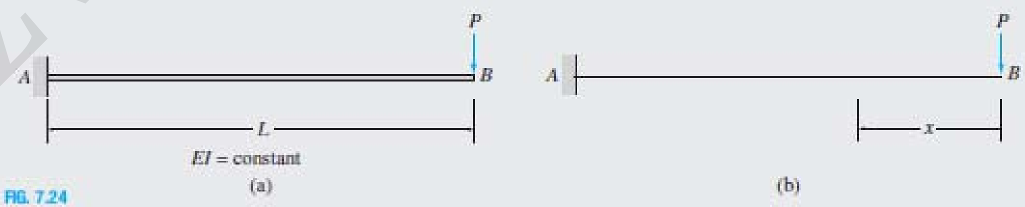


FIG. 7.24

Solution

Using the x coordinate shown in Fig. 7.24(b), we write the equation for the bending moment in the beam as

$$M = -Px$$

The partial derivative of M with respect to P is given by

$$\frac{\partial M}{\partial P} = -x$$

continued

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The deflection at B can now be obtained by applying the expression of Castigliano's second theorem, as given by Eq. (7.60), as follows:

$$\Delta_B = \int_0^L \left(\frac{\partial M}{\partial P}\right)\left(\frac{M}{EI}\right) dx$$

$$\Delta_B = \int_0^L (-x)\left(-\frac{Px}{EI}\right) dx$$

$$= \frac{P}{EI}\int_0^L x^2\, dx = \frac{PL^3}{3EI}$$

$$\Delta_B = \frac{PL^3}{3EI} \downarrow$$

Ans.

Example 7.17

Determine the rotation of joint C of the frame shown in Fig. 7.25(a) by Castigliano's second theorem.

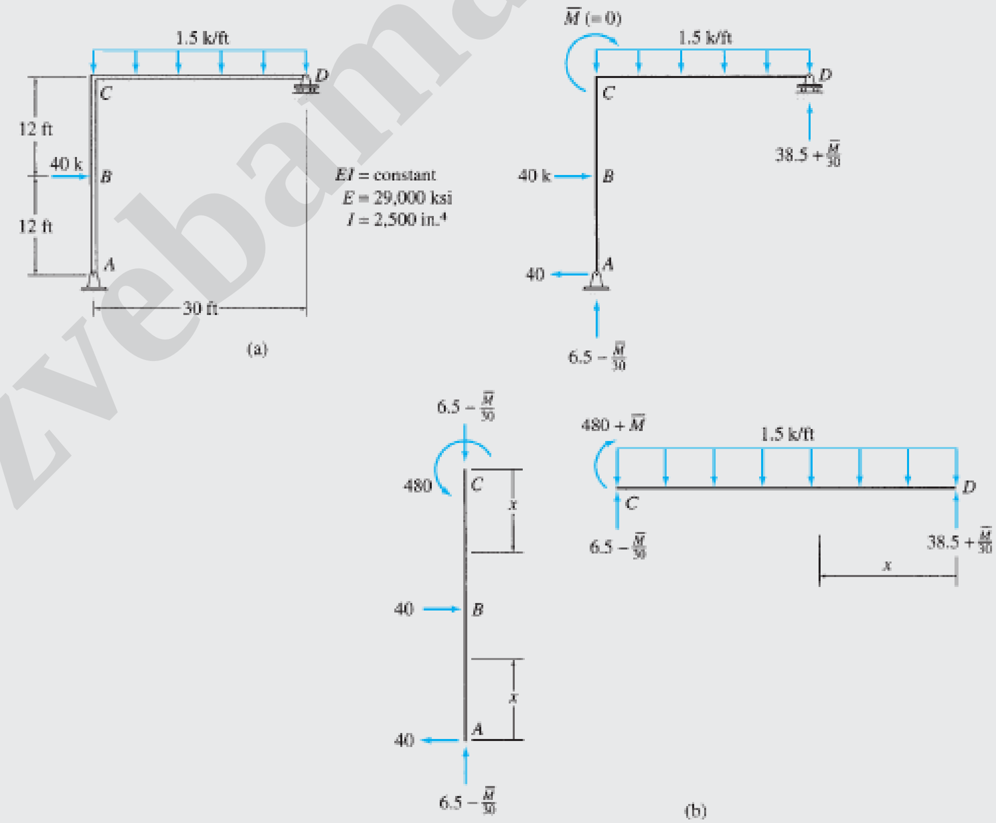

FIG. 7.25

continued

Solution

This frame was previously analyzed by the virtual work method in Example 7.11.

No external couple is acting at joint C, where the rotation is desired, so we apply a fictitious couple $\bar{M}$ $(=0)$ at C, as shown in Fig. 7.25(b). The x coordinates used for determining the bending moment equations for the three segments of the frame are also shown in Fig. 7.25(b), and the equations for M in terms of $\bar{M}$ and $\partial M/\partial \bar{M}$ obtained for the three segments are tabulated in Table 7.13. The rotation of joint C of the frame can now be determined by setting $\bar{M}=0$ in the equations for M and $\partial M/\partial \bar{M}$ and by applying the expression of Castigliano's second theorem as given by Eq. (7.65):

$$\theta_C = \sum \int \left(\frac{\partial M}{\partial \bar{M}}\right)\frac{M}{EI}dx$$

$$= \int_0^{30}\left(\frac{x}{30}\right)\left(38.5x - 1.5\frac{x^2}{2}\right)dx$$

$$= \frac{6{,}487.5 \text{ k-ft}^2}{EI} = \frac{6{,}487.5(12)^2}{(29{,}000)(2{,}500)} = 0.0129 \text{ rad}$$

$$\theta_C = 0.0129 \text{ rad}$$ **Ans.**

TABLE 7.13

Segment	x Coordinate: Origin	x Coordinate: Limits (ft)	M (k-ft)	$\frac{\partial M}{\partial \bar{M}}$ (k-ft/k-ft)
AB	A	0–12	$40x$	0
CB	C	0–12	480	0
DC	D	0–30	$\left(38.5+\frac{\bar{M}}{30}\right)x - 1.5\frac{x^2}{2}$	$\frac{x}{30}$

Example 7.18

Use Castigliano's second theorem to determine the horizontal and vertical components of the deflection at joint B of the truss shown in Fig. 7.26(a).

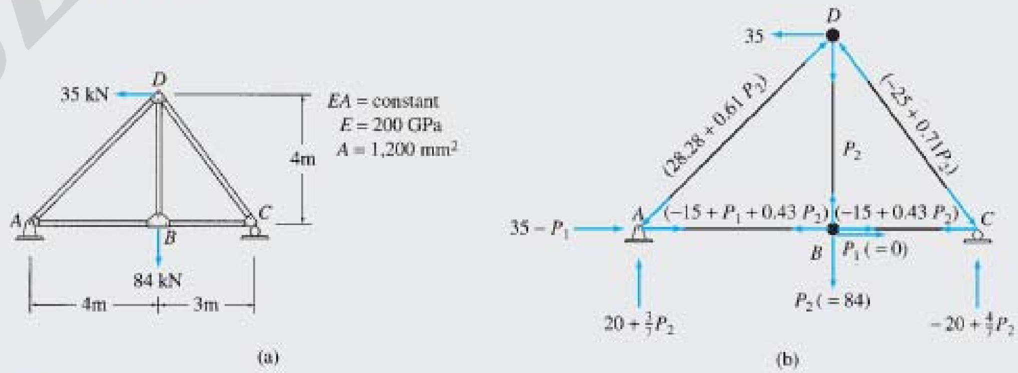

FIG. 7.26

continued

Solution

This truss was previously analyzed by the virtual work method in Example 7.3.

TABLE 7.14

Member	L (m)	F (kN)	$\frac{\partial F}{\partial P_1}$ (kN/kN)	$\frac{\partial F}{\partial P_2}$ (kN/kN)	For $P_1 = 0$ and $P_2 = 84$ kN: $(\partial F/\partial P_1)FL$ (kN · m)	$(\partial F/\partial P_2)FL$ (kN · m)
AB	4	$-15 + P_1 + 0.43P_2$	1	0.43	84.48	36.32
BC	3	$-15 + 0.43P_2$	0	0.43	0	27.24
AD	5.66	$-28.28 - 0.61P_2$	0	−0.61	0	274.55
BD	4	P_2	0	1	0	336.00
CD	5	$25 - 0.71P_2$	0	−0.71	0	122.97
				$\sum\left(\frac{\partial F}{\partial P}\right)FL$	84.48	797.08

$$\Delta_{BH} = \frac{1}{EA}\sum\left(\frac{\partial F}{\partial P_1}\right)FL$$

$$= \frac{84.48}{EA}\ \text{kN} \cdot \text{m}$$

$$= \frac{84.48}{200(10^6)(0.0012)} = 0.00035\ \text{m}$$

$$\Delta_{BH} = 0.35\ \text{mm} \rightarrow \qquad \textbf{Ans.}$$

$$\Delta_{BV} = \frac{1}{EA}\sum\left(\frac{\partial F}{\partial P_2}\right)FL$$

$$= \frac{797.08}{EA}\ \text{kN} \cdot \text{m}$$

$$= \frac{797.08}{200(10^6)(0.0012)} = 0.00332\ \text{m}$$

$$\Delta_{BV} = 3.32\ \text{mm} \downarrow \qquad \textbf{Ans.}$$

As shown in Fig. 7.26(b), a fictitious horizontal force P_1 (= 0) is applied at joint *B* to determine the horizontal component of deflection, whereas the 84-kN vertical load is designated as the variable P_2 to be used for computing the vertical component of deflection at joint *B*. The member axial forces, in terms of P_1 and P_2, are then determined by applying the method of joints. These member forces *F*, along with their partial derivatives with respect to P_1 and P_2, are tabulated in Table 7.14. Note that the tensile axial forces are considered as positive and the compressive forces are negative. Numerical values of $P_1 = 0$ and $P_2 = 84$ kN are then substituted in the equations for *F*, and the expression of Castigliano's second theorem, as given by Eq. (7.59) is applied, as shown in the table, to determine the horizontal and vertical components of the deflection at joint *B* of the truss.

7.8 Betti's Law and Maxwell's Law of Reciprocal Deflections

Maxwell's law of reciprocal deflections, initially developed by James C. Maxwell in 1864, plays an important role in the analysis of statically indeterminate structures to be considered in Part Three of this text. Maxwell's law will be derived here as a special case of the more general *Betti's law*, which was presented by E. Betti in 1872. Betti's law can be stated as follows:

> For a linearly elastic structure, the virtual work done by a P system of forces and couples acting through the deformation caused by a Q system of forces and couples is equal to the virtual work of the Q system acting through the deformation due to the P system.

To show the validity of this law, consider the beam shown in Fig. 7.27. The beam is subjected to two different systems of forces, P and Q systems, as shown in Fig. 7.27(a) and (b), respectively. Now, let us assume that we

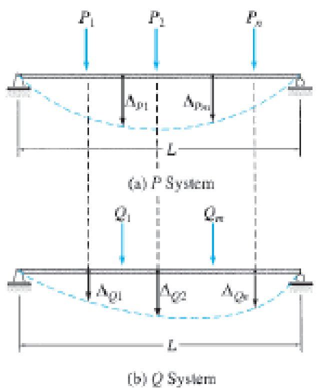

FIG. 7.27

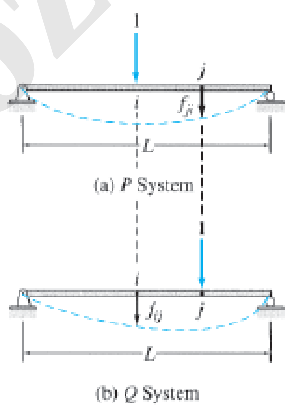

FIG. 7.28

subject the beam that has the P forces already acting on it (Fig. 7.27(a)) to the deflections caused by the Q system of forces (Fig. 7.27(b)). The virtual external work (W_{ve}) done can be written as

$$W_{ve} = P_1\Delta_{Q1} + P_2\Delta_{Q2} + \cdots + P_n\Delta_{Qn}$$

or

$$W_{ve} = \sum_{i=1}^{n} P_i\Delta_{Qi}$$

By applying the principle of virtual forces for deformable bodies, $W_{ve} = W_{vi}$, and using the expression for the virtual internal work done in beams (Eq. (7.29)), we obtain

$$\sum_{i=1}^{n} P_i\Delta_{Qi} = \int_0^L \frac{M_P M_Q}{EI}dx \qquad (7.66)$$

Next, we assume that the beam with the Q forces acting on it (Fig. 7.27(b)) is subjected to the deflections caused by the P forces (Fig. 7.27(a)). By equating the virtual external work to the virtual internal work, we obtain

$$\sum_{j=1}^{m} Q_j\Delta_{Pj} = \int_0^L \frac{M_Q M_P}{EI}dx \qquad (7.67)$$

Noting that the right-hand sides of Eqs. (7.66) and (7.67) are identical, we equate the left-hand sides to obtain

$$\sum_{i=1}^{n} P_i\Delta_{Qi} = \sum_{j=1}^{m} Q_j\Delta_{Pj} \qquad (7.68)$$

Equation (7.68) represents the mathematical statement of Betti's law.

Maxwell's law of reciprocal deflections states that *for a linearly elastic structure, the deflection at a point i due to a unit load applied at a point j is equal to the deflection at j due to a unit load at i.*

In this statement, the terms *deflection* and *load* are used in the general sense to include rotation and couple, respectively. As mentioned previously, Maxwell's law can be considered as a special case of Betti's law. To prove Maxwell's law, consider the beam shown in Fig. 7.28. The beam is separately subjected to the P and Q systems, consisting of the unit loads at points i and j, respectively, as shown in Fig. 7.28(a) and (b). As the figure indicates, f_{ij} represents the deflection at i due to the unit load at j, whereas f_{ji} denotes the deflection at j due to the unit load at i. These deflections per unit load are referred to as *flexibility coefficients*. By applying Betti's law (Eq. (7.68)), we obtain

$$1(f_{ij}) = 1(f_{ji})$$

or

$$f_{ij} = f_{ji} \qquad (7.69)$$

which is the mathematical statement of Maxwell's law.

The reciprocal relationship remains valid between the rotations caused by two unit couples as well as between the deflection and the rotation caused by a unit couple and a unit force, respectively.

Summary

In this chapter we have learned that the work done by a force P (or couple M) during a displacement Δ (or rotation θ) of its point of application in the direction of its line of action is given by

$$W = \int_0^{\Delta} P \, d\Delta \tag{7.1}$$

or

$$W = \int_0^{\theta} M \, d\theta \tag{7.4}$$

The principle of virtual work for rigid bodies states that if a rigid body is in equilibrium under a system of forces and if it is subjected to any small virtual rigid-body displacement, the virtual work done by the external forces is zero.

The principle of virtual forces for deformable bodies can be mathematically stated as

$$W_{ve} = W_{vi} \tag{7.16}$$

in which W_{ve} = virtual external work done by virtual external forces (and couples) acting through the real external displacements (and rotations) of the structure; and W_{vi} = virtual internal work done by the virtual internal forces (and couples) acting through the real internal displacements (and rotations) of the structure.

The method of virtual work for determining the deformations of structures is based on the principle of virtual forces for deformable bodies. The method employs two separate systems: (1) a real system of loads (or other effects) causing the deformation to be determined and (2) a virtual system consisting of a unit load (or unit couple) applied at the point and in the direction of the desired deflection (or rotation). The explicit expressions of the virtual work method to be used to determine the deflections of trusses, beams, and frames are as follows:

$$\text{Trusses} \quad 1(\Delta) = \sum F_v \left(\frac{FL}{AE} \right) \tag{7.23}$$

$$\text{Beams} \quad 1(\Delta) = \int_0^L \frac{M_v M}{EI} dx \tag{7.30}$$

$$\text{Frames} \quad 1(\Delta) = \sum F_v \left(\frac{FL}{AE} \right) + \sum \int \frac{M_v M}{EI} dx \tag{7.35}$$

The principle of conservation of energy states that the work performed by statically applied external forces on an elastic structure in equilibrium is equal to the work done by internal forces or the strain energy stored in the structure. The expressions for the strain energy of trusses, beams and frames are

$$\text{Trusses} \quad U = \sum \frac{F^2 L}{2AE} \tag{7.41}$$

$$\text{Beams} \quad U = \int_0^L \frac{M^2}{2EI} dx \tag{7.44}$$

$$\text{Frames} \quad U = \sum \frac{F^2 L}{2AE} + \sum \int \frac{M^2}{2EI} dx \tag{7.48}$$

Castigliano's second theorem for linearly elastic structures can be mathematically expressed as

$$\frac{\partial U}{\partial P_i} = \Delta_i \quad \text{or} \quad \frac{\partial U}{\partial M_i} = \theta_i \tag{7.50}$$

The expressions of Castigliano's second theorem, which can be used to determine deflections, are as follows:

$$\text{Trusses} \quad \Delta = \sum \left(\frac{\partial F}{\partial P}\right) \frac{FL}{AE} \tag{7.59}$$

$$\text{Beams} \quad \Delta = \int_0^L \left(\frac{\partial M}{\partial P}\right) \frac{M}{EI} dx \tag{7.60}$$

$$\text{Frames} \quad \Delta = \sum \left(\frac{\partial F}{\partial P}\right) \frac{FL}{AE} + \sum \int \left(\frac{\partial M}{\partial P}\right) \frac{M}{EI} dx \tag{7.62}$$

Maxwell's law of reciprocal deflections states that, for a linearly elastic structure, the deflection at a point i due to a unit load applied at a point j is equal to the deflection at j due to a unit load at i.