تغییر شکل تیر

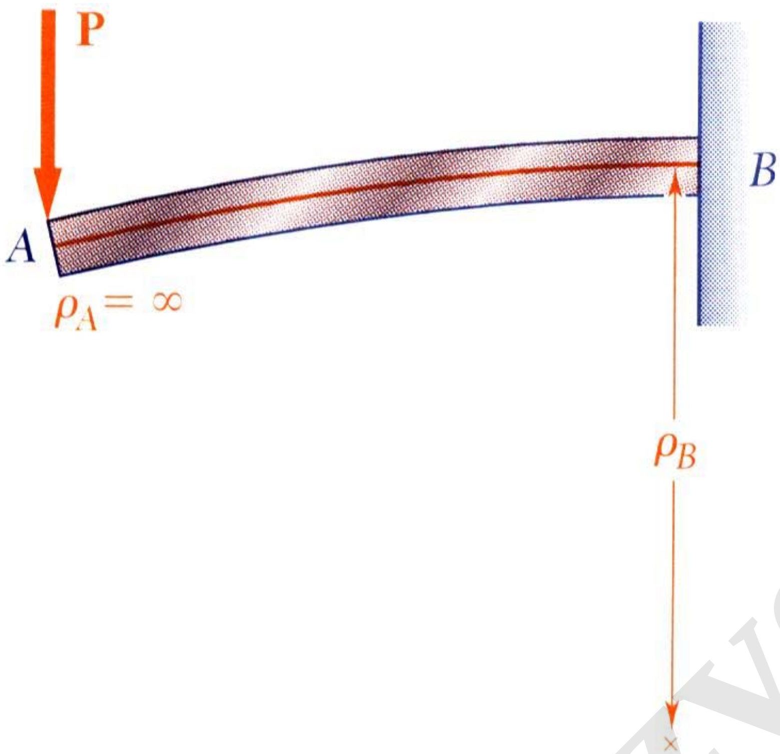
Deflection of Beams

رضا عباسی- دانشگاه آزاد اسلامی واحد تهران جنوب

معادلات تغییر شکل تیر

روش انتگرال گیری – روش مستقیم

Deformation of a Beam Under Transverse Loading



- Relationship between bending moment and curvature for pure bending remains valid for general transverse loadings.

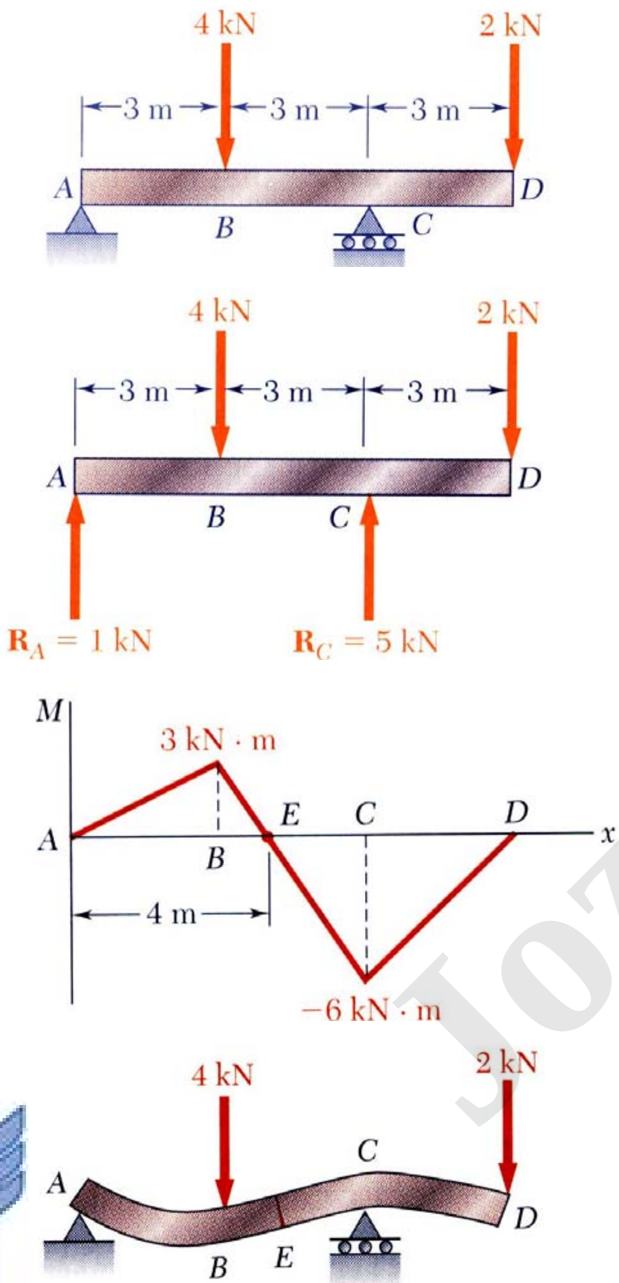
$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

- Cantilever beam subjected to concentrated load at the free end,

$$\frac{1}{\rho} = -\frac{Px}{EI}$$

- Curvature varies linearly with x
- At the free end A, $\frac{1}{\rho_A} = 0, \quad \rho_A = \infty$
- At the support B, $\frac{1}{\rho_B} \neq 0, \quad |\rho_B| = \frac{EI}{PL}$

Deformation of a Beam Under Transverse Loading

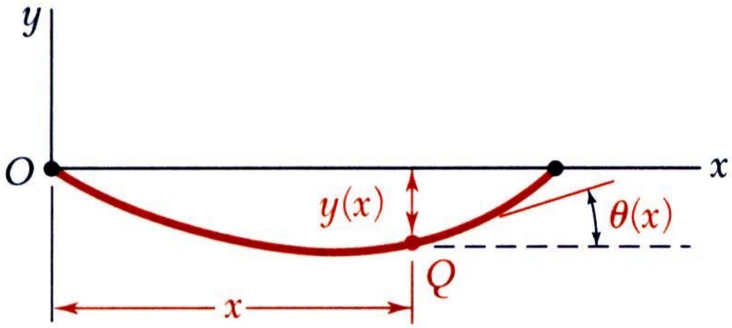


- Overhanging beam
- Reactions at *A* and *C*
- Bending moment diagram
- Curvature is zero at points where the bending moment is zero, i.e., at each end and at *E*.

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

- Beam is concave upwards where the bending moment is positive and concave downwards where it is negative.
- Maximum curvature occurs where the moment magnitude is a maximum.
- An equation for the beam shape or *elastic curve* is required to determine maximum deflection and slope.

Equation of the Elastic Curve



- From elementary calculus, simplified for beam parameters,

$$\frac{1}{\rho} = \frac{\dfrac{d^2 y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} \approx \frac{d^2 y}{dx^2}$$

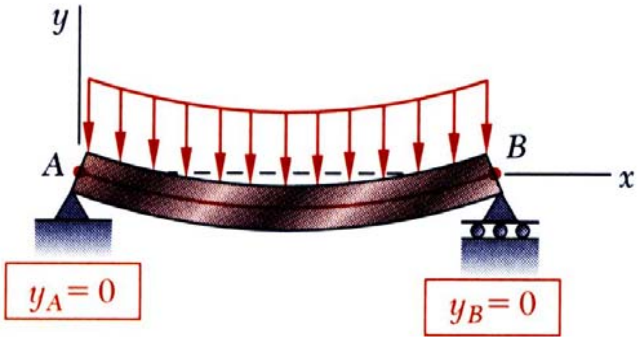
- Substituting and integrating,

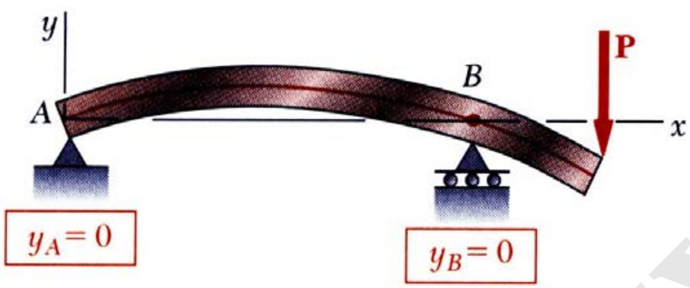
$$EI\frac{1}{\rho} = EI\frac{d^2 y}{dx^2} = M(x)$$

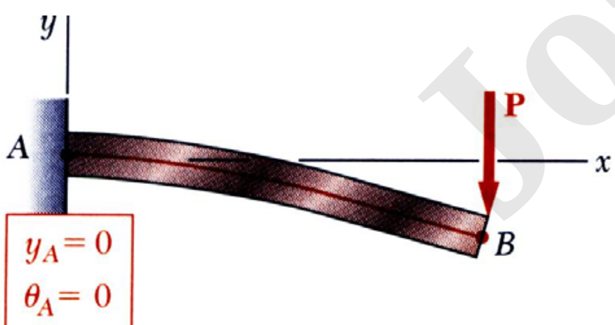
$$EI\,\theta \approx EI\frac{dy}{dx} = \int_0^x M(x)dx + C_1$$

$$EI\,y = \int_0^x dx \int_0^x M(x)dx + C_1 x + C_2$$

Equation of the Elastic Curve







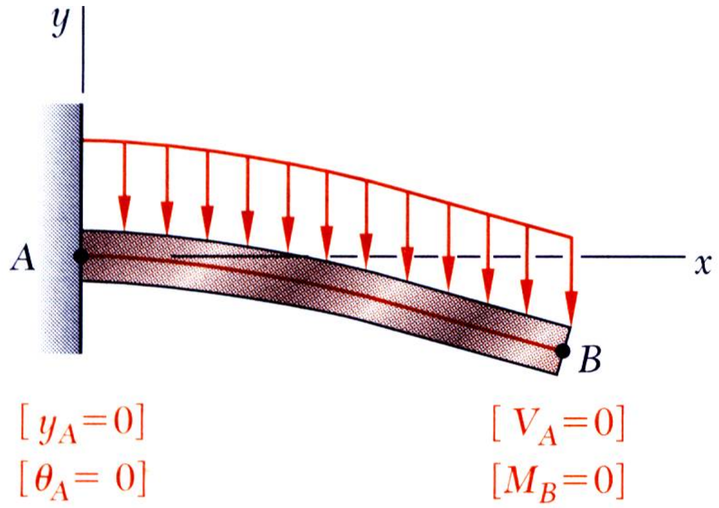
- Constants are determined from boundary conditions

$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1 x + C_2$$

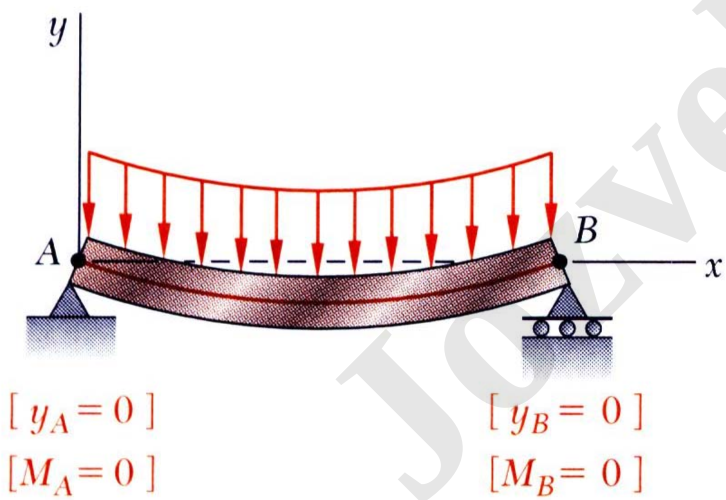
- Three cases for statically determinant beams,
  - Simply supported beam
    $y_A = 0, \quad y_B = 0$
  - Overhanging beam
    $y_A = 0, \quad y_B = 0$
  - Cantilever beam
    $y_A = 0, \quad \theta_A = 0$

- More complicated loadings require multiple integrals and application of requirement for continuity of displacement and slope.

Direct Determination of the Elastic Curve From the Load Distribution



(a) Cantilever beam



(b) Simply supported beam

- For a beam subjected to a distributed load,

$$\frac{dM}{dx}=V(x) \qquad \frac{d^2M}{dx^2}=\frac{dV}{dx}=-w(x)$$

- Equation for beam displacement becomes

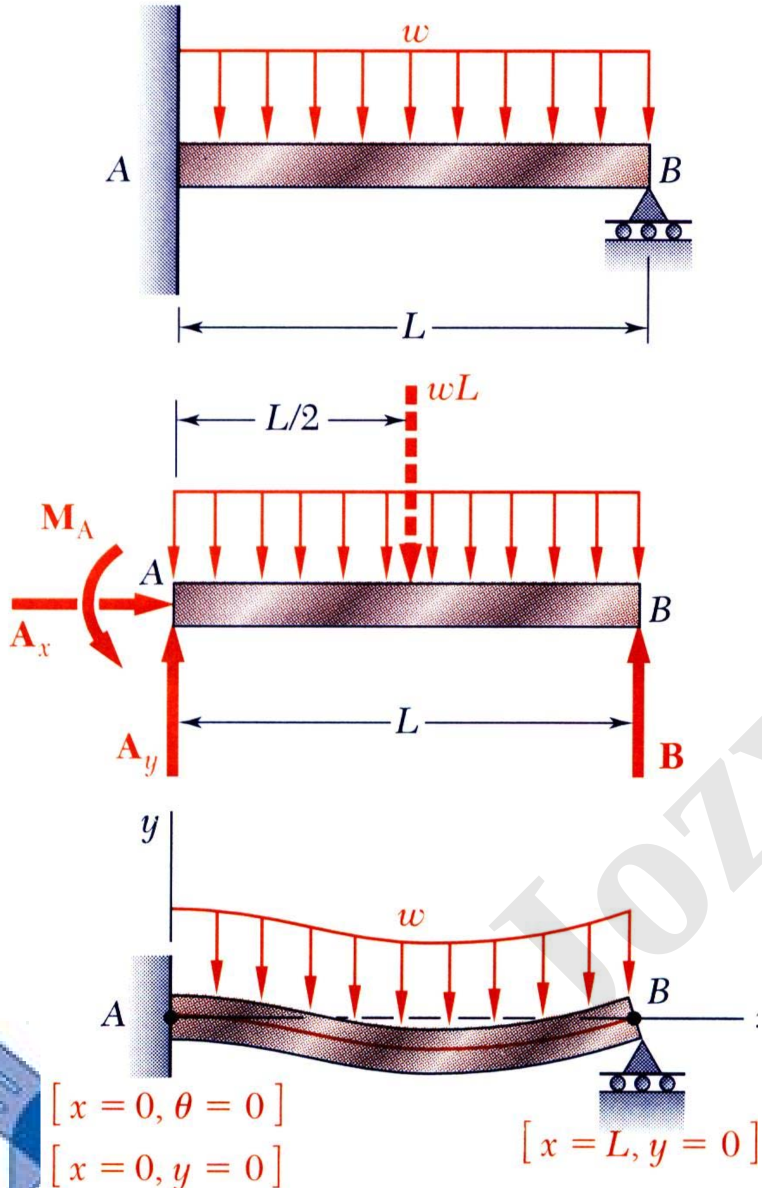
$$\frac{d^2M}{dx^2}=EI\frac{d^4y}{dx^4}=-w(x)$$

- Integrating four times yields

$$EI\,y(x)=-\int dx\int dx\int dx\int w(x)dx$$
$$+\tfrac{1}{6}C_1x^3+\tfrac{1}{2}C_2x^2+C_3x+C_4$$

- Constants are determined from boundary conditions.

Statically Indeterminate Beams



- Consider beam with fixed support at A and roller support at B.
- From free-body diagram, note that there are four unknown reaction components.
- Conditions for static equilibrium yield

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_A = 0$$

The beam is statically indeterminate.

- Also have the beam deflection equation,

$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1 x + C_2$$

which introduces two unknowns but provides three additional equations from the boundary conditions:

$$\text{At } x = 0,\ \theta = 0\ y = 0 \qquad \text{At } x = L,\ y = 0$$

EXAMPLE 9.03

For the prismatic beam and the loading shown (Fig. 9.16), determine the slope and deflection at point D.

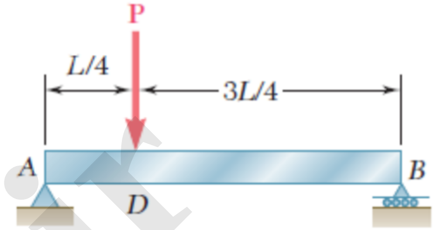


Fig. 9.16

We must divide the beam into two portions, AD and DB, and determine the function $y(x)$ which defines the elastic curve for each of these portions.

1. From A to D ($x < L/4$). We draw the free-body diagram of a portion of beam AE of length $x < L/4$ (Fig. 9.17). Taking moments about E, we have

$$M_1 = \frac{3P}{4}x \tag{9.17}$$

or, recalling Eq. (9.4),

$$EI\frac{d^2y_1}{dx^2} = \frac{3}{4}Px \tag{9.18}$$

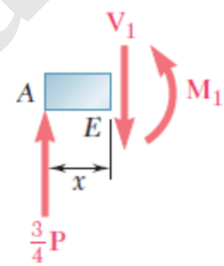


Fig. 9.17

where $y_1(x)$ is the function which defines the elastic curve *for portion AD of the beam*. Integrating in x, we write

$$EI\,\theta_1 = EI\frac{dy_1}{dx} = \frac{3}{8}Px^2 + C_1 \tag{9.19}$$

$$EI\,y_1 = \frac{1}{8}Px^3 + C_1x + C_2 \tag{9.20}$$

2. From D to B ($x > L/4$). We now draw the free-body diagram of a portion of beam AE of length $x > L/4$ (Fig. 9.18) and write

$$M_2 = \frac{3P}{4}x - P\left(x - \frac{L}{4}\right) \tag{9.21}$$

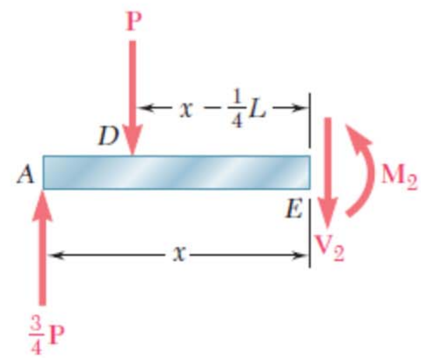


Fig. 9.18

or, recalling Eq. (9.4) and rearranging terms,

$$EI\frac{d^2y_2}{dx^2} = -\frac{1}{4}Px + \frac{1}{4}PL \tag{9.22}$$

where $y_2(x)$ is the function which defines the elastic curve *for portion DB of the beam*. Integrating in x, we write

$$EI\,\theta_2 = EI\frac{dy_2}{dx} = -\frac{1}{8}Px^2 + \frac{1}{4}PLx + C_3 \tag{9.23}$$

$$EI\,y_2 = -\frac{1}{24}Px^3 + \frac{1}{8}PLx^2 + C_3x + C_4 \tag{9.24}$$

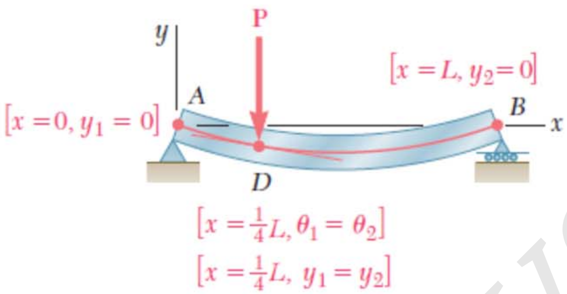


Fig. 9.19

Determination of the Constants of Integration. The conditions that must be satisfied by the constants of integration have been summarized in Fig. 9.19. At the support A, where the deflection is defined by Eq. (9.20), we must have $x = 0$ and $y_1 = 0$. At the support B, where the deflection is defined by Eq. (9.24), we must have $x = L$ and $y_2 = 0$. Also, the fact that there can be no sudden change in deflection or in slope at point D requires that $y_1 = y_2$ and $\theta_1 = \theta_2$ when $x = L/4$. We have therefore:

$[x = 0, y_1 = 0]$, Eq. (9.20): $\quad 0 = C_2 \qquad (9.25)$

$[x = L, y_2 = 0]$, Eq. (9.24): $\quad 0 = \frac{1}{12}PL^3 + C_3L + C_4 \qquad (9.26)$

$[x = L/4, \theta_1 = \theta_2]$, Eqs. (9.19) and (9.23):

$$\frac{3}{128}PL^2 + C_1 = \frac{7}{128}PL^2 + C_3 \tag{9.27}$$

$[x = L/4, y_1 = y_2]$, Eqs. (9.20) and (9.24):

$$\frac{PL^3}{512} + C_1\frac{L}{4} = \frac{11PL^3}{1536} + C_3\frac{L}{4} + C_4 \tag{9.28}$$

Solving these equations simultaneously, we find

$$C_1 = -\frac{7PL^2}{128}, C_2 = 0, C_3 = -\frac{11PL^2}{128}, C_4 = \frac{PL^3}{384}$$

Substituting for C_1 and C_2 into Eqs. (9.19) and (9.20), we write that for $x \leq L/4$,

$$EI\,\theta_1 = \frac{3}{8}Px^2 - \frac{7PL^2}{128} \tag{9.29}$$

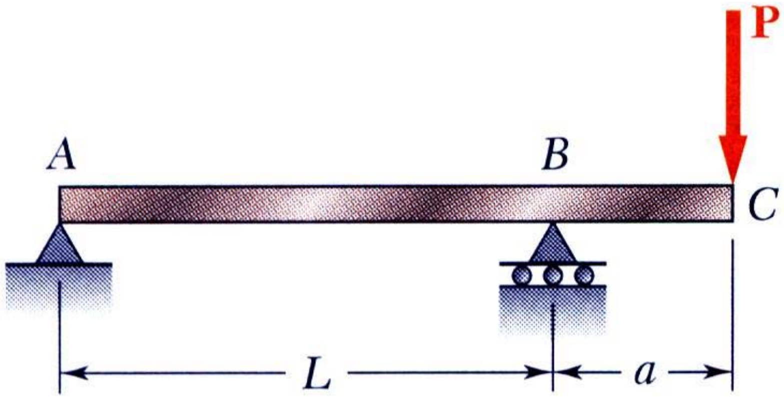
$$EI\,y_1 = \frac{1}{8}Px^3 - \frac{7PL^2}{128}x \tag{9.30}$$

Letting $x = L/4$ in each of these equations, we find that the slope and deflection at point D are, respectively,

$$\theta_D = -\frac{PL^2}{32EI} \quad \text{and} \quad y_D = -\frac{3PL^3}{256EI}$$

We note that, since $\theta_D \neq 0$, the deflection at D is *not* the maximum deflection of the beam.

Sample Problem 9.1



$W14 \times 68$ $\quad I = 723\,\text{in}^4$ $\quad E = 29 \times 10^6\,\text{psi}$

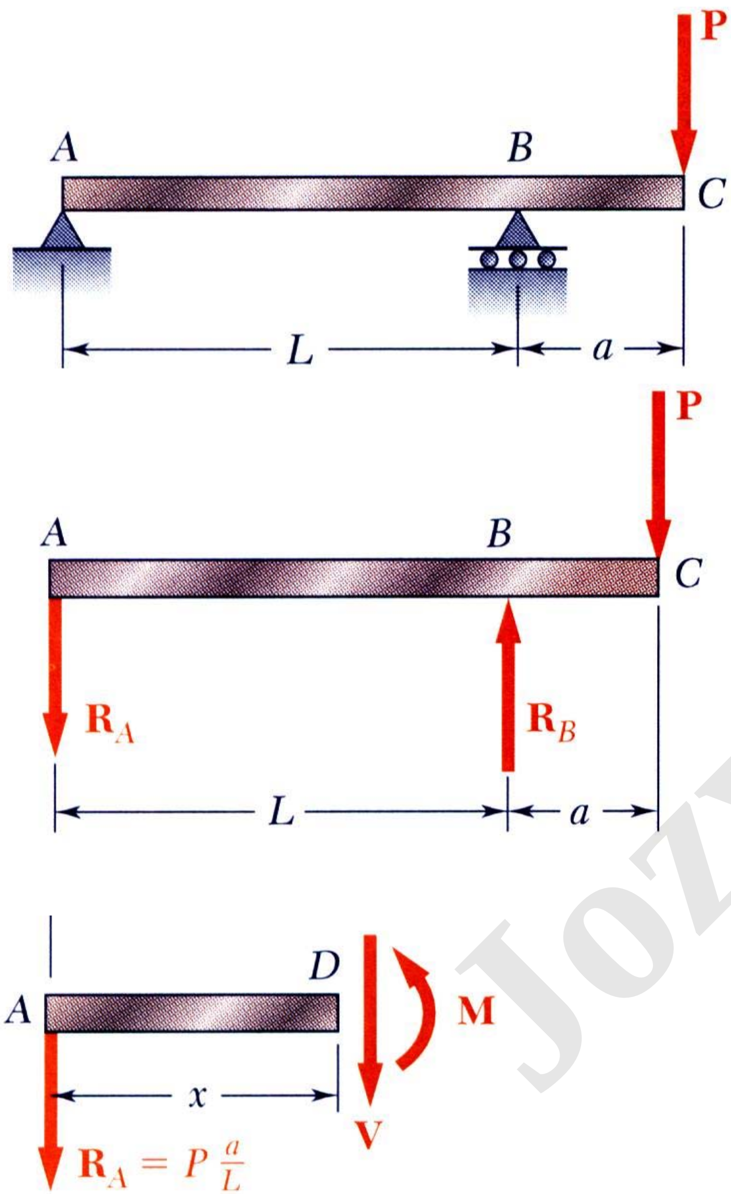
$P = 50\,\text{kips}$ $\quad L = 15\,\text{ft}$ $\quad a = 4\,\text{ft}$

For portion *AB* of the overhanging beam, (*a*) derive the equation for the elastic curve, (*b*) determine the maximum deflection,
(*c*) evaluate y_{max}.

SOLUTION:

- Develop an expression for M(x) and derive differential equation for elastic curve.
- Integrate differential equation twice and apply boundary conditions to obtain elastic curve.
- Locate point of zero slope or point of maximum deflection.
- Evaluate corresponding maximum deflection.

Sample Problem 9.1



SOLUTION:

- Develop an expression for M(x) and derive differential equation for elastic curve.

  - Reactions:

$$R_A = \frac{Pa}{L} \downarrow \quad R_B = P\left(1 + \frac{a}{L}\right) \uparrow$$

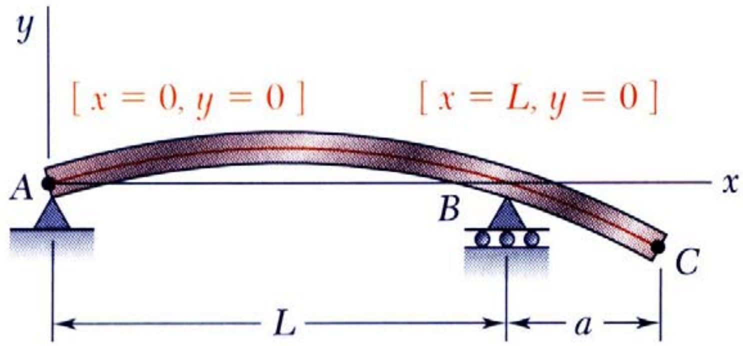
  - From the free-body diagram for section *AD*,

$$M = -P\frac{a}{L}x \quad (0 < x < L)$$

  - The differential equation for the elastic curve,

$$EI\frac{d^2y}{dx^2} = -P\frac{a}{L}x$$

Sample Problem 9.1



$$EI\frac{d^2y}{dx^2} = -P\frac{a}{L}x$$

- Integrate differential equation twice and apply boundary conditions to obtain elastic curve.

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + C_1$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + C_1x + C_2$$

at $x = 0,\ y = 0:\quad C_2 = 0$

at $x = L,\ y = 0:\quad 0 = -\frac{1}{6}P\frac{a}{L}L^3 + C_1L \qquad C_1 = \frac{1}{6}PaL$

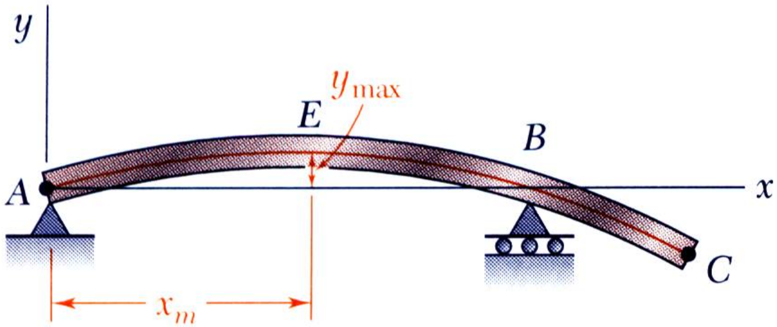
Substituting,

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + \frac{1}{6}PaL \qquad \frac{dy}{dx} = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x}{L}\right)^2\right]$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + \frac{1}{6}PaLx$$

$$y = \frac{PaL^2}{6EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^3\right]$$

Sample Problem 9.1



$$y = \frac{PaL^2}{6EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^3\right]$$

- Locate point of zero slope or point of maximum deflection.

$$\frac{dy}{dx} = 0 = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x_m}{L}\right)^2\right] \qquad x_m = \frac{L}{\sqrt{3}} = 0.577L$$

- Evaluate corresponding maximum deflection.

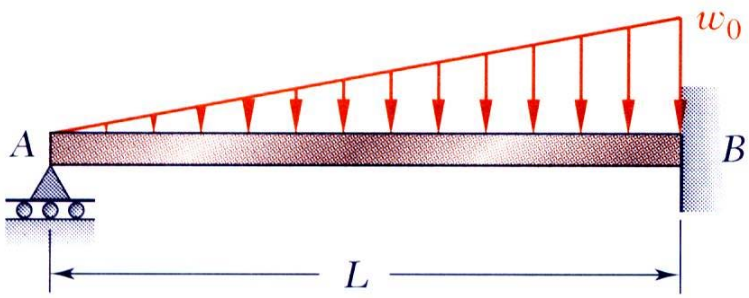
$$y_{max} = \frac{PaL^2}{6EI}\left[0.577 - (0.577)^3\right]$$

$$y_{max} = 0.0642\frac{PaL^2}{6EI}$$

$$y_{max} = 0.0642\frac{(50\,\text{kips})(48\,\text{in})(180\,\text{in})^2}{6\left(29\times10^6\,\text{psi}\right)\left(723\,\text{in}^4\right)}$$

$$y_{max} = 0.238\,\text{in}$$

Sample Problem 9.3

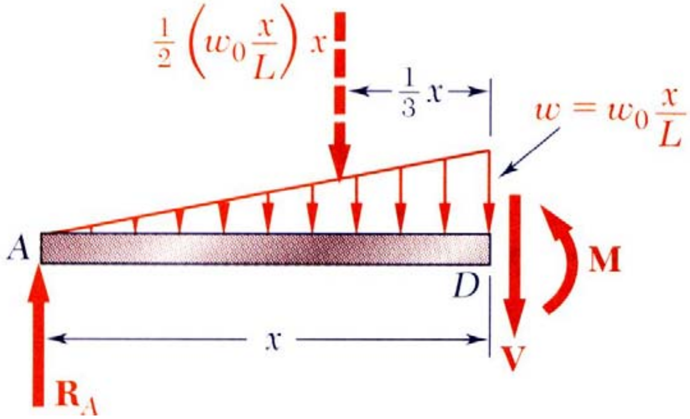


For the uniform beam, determine the reaction at *A*, derive the equation for the elastic curve, and determine the slope at *A*. (Note that the beam is statically indeterminate to the first degree)

SOLUTION:

- Develop the differential equation for the elastic curve (will be functionally dependent on the reaction at *A*).
- Integrate twice and apply boundary conditions to solve for reaction at *A* and to obtain the elastic curve.
- Evaluate the slope at *A*.

Sample Problem 9.3



- Consider moment acting at section D,

$$\sum M_D = 0$$

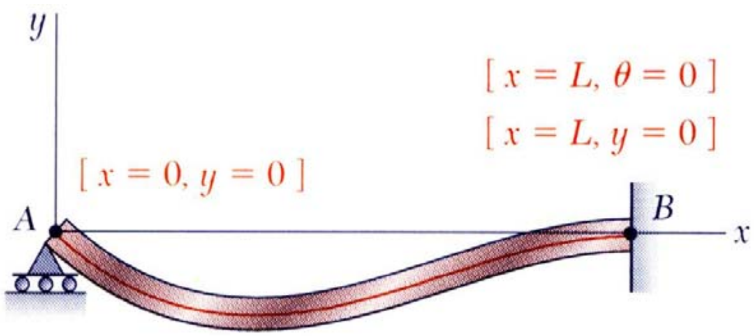
$$R_A x - \frac{1}{2}\left(\frac{w_0 x^2}{L}\right)\frac{x}{3} - M = 0$$

$$M = R_A x - \frac{w_0 x^3}{6L}$$

- The differential equation for the elastic curve,

$$EI\frac{d^2 y}{dx^2} = M = R_A x - \frac{w_0 x^3}{6L}$$

Sample Problem 9.3



$$EI\frac{d^2y}{dx^2} = M = R_A x - \frac{w_0 x^3}{6L}$$

- Integrate twice

$$EI\frac{dy}{dx} = EI\theta = \frac{1}{2}R_A x^2 - \frac{w_0 x^4}{24L} + C_1$$

$$EI\,y = \frac{1}{6}R_A x^3 - \frac{w_0 x^5}{120L} + C_1 x + C_2$$

- Apply boundary conditions:

at $x = 0,\ y = 0:\quad C_2 = 0$

at $x = L,\ \theta = 0:\quad \frac{1}{2}R_A L^2 - \frac{w_0 L^3}{24} + C_1 = 0$

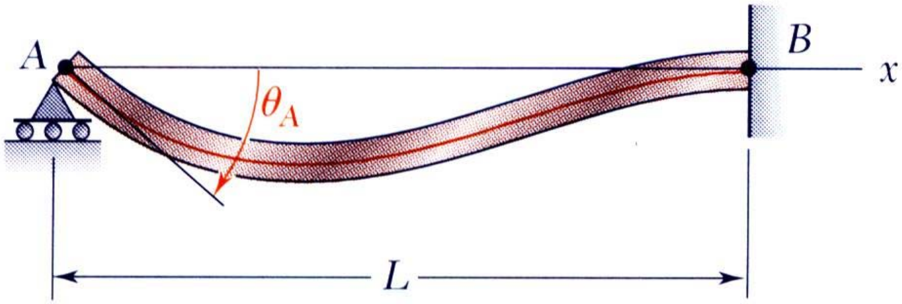
at $x = L,\ y = 0:\quad \frac{1}{6}R_A L^3 - \frac{w_0 L^4}{120} + C_1 L + C_2 = 0$

- Solve for reaction at A

$$\frac{1}{3}R_A L^3 - \frac{1}{30}w_0 L^4 = 0$$

$$R_A = \frac{1}{10}w_0 L \uparrow$$

Sample Problem 9.3



- Substitute for C_1, C_2, and R_A in the elastic curve equation,

$$EI\,y = \frac{1}{6}\left(\frac{1}{10}w_0 L\right)x^3 - \frac{w_0 x^5}{120L} - \left(\frac{1}{120}w_0 L^3\right)x$$

$$y = \frac{w_0}{120EIL}\left(-x^5 + 2L^2x^3 - L^4x\right)$$

- Differentiate once to find the slope,

$$\theta = \frac{dy}{dx} = \frac{w_0}{120EIL}\left(-5x^4 + 6L^2x^2 - L^4\right)$$

at $x = 0$, $$\theta_A = \frac{w_0 L^3}{120EI}$$

EXAMPLE 9.04

The simply supported prismatic beam AB carries a uniformly distributed load w per unit length (Fig. 9.21). Determine the equation of the elastic curve and the maximum deflection of the beam. (This is the same beam and loading as in Example 9.02.)

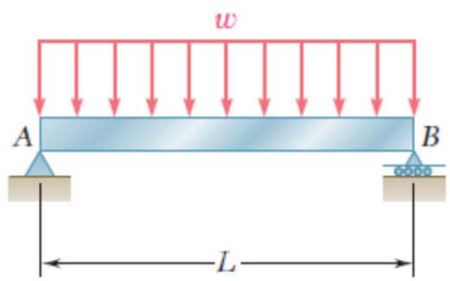


Fig. 9.21

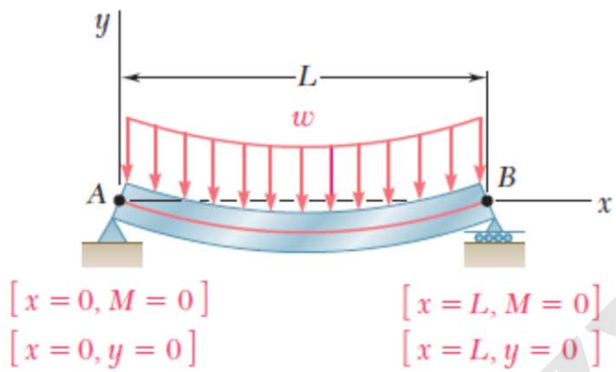


Fig. 9.22

Since w = constant, the first three of Eqs. (9.33) yield

$$EI\frac{d^4y}{dx^4} = -w$$

$$EI\frac{d^3y}{dx^3} = V(x) = -wx + C_1$$

$$EI\frac{d^2y}{dx^2} = M(x) = -\frac{1}{2}wx^2 + C_1x + C_2 \qquad (9.34)$$

Noting that the boundary conditions require that $M = 0$ at both ends of the beam (Fig. 9.22), we first let $x = 0$ and $M = 0$ in Eq. (9.34) and obtain $C_2 = 0$. We then make $x = L$ and $M = 0$ in the same equation and obtain $C_1 = \frac{1}{2}wL$.

Carrying the values of C_1 and C_2 back into Eq. (9.34), and integrating twice, we write

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx$$

$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_3$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 + C_3x + C_4 \qquad (9.35)$$

But the boundary conditions also require that $y = 0$ at both ends of the beam. Letting $x = 0$ and $y = 0$ in Eq. (9.35), we obtain $C_4 = 0$; letting $x = L$ and $y = 0$ in the same equation, we write

$$0 = -\tfrac{1}{24}wL^4 + \tfrac{1}{12}wL^4 + C_3L$$
$$C_3 = -\tfrac{1}{24}wL^3$$

Carrying the values of C_3 and C_4 back into Eq. (9.35) and dividing both members by EI, we obtain the equation of the elastic curve:

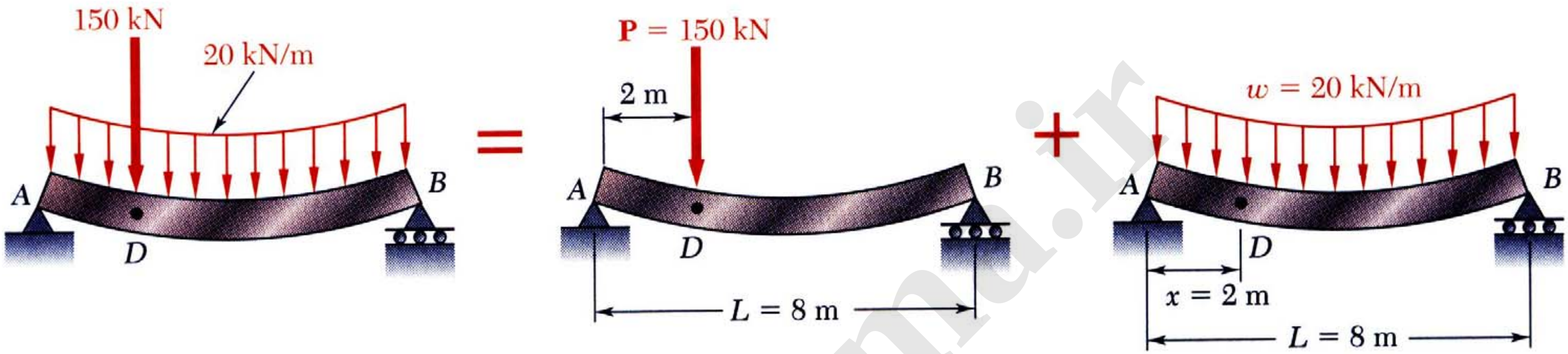
$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x) \qquad (9.36)$$

The value of the maximum deflection is obtained by making $x = L/2$ in Eq. (9.36). We have

$$|y|_{\max} = \frac{5wL^4}{384EI}$$

تحلیل تیرهای نامعین
با استفاده از معادلات تغییر مکان تیر
روش نیرو

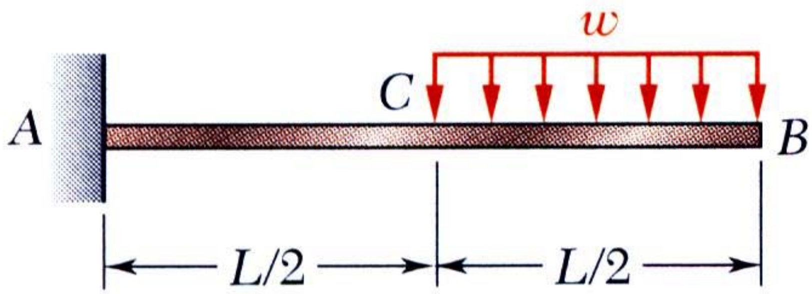
Method of Superposition



Principle of Superposition:

- Deformations of beams subjected to combinations of loadings may be obtained as the linear combination of the deformations from the individual loadings
- Procedure is facilitated by tables of solutions for common types of loadings and supports.

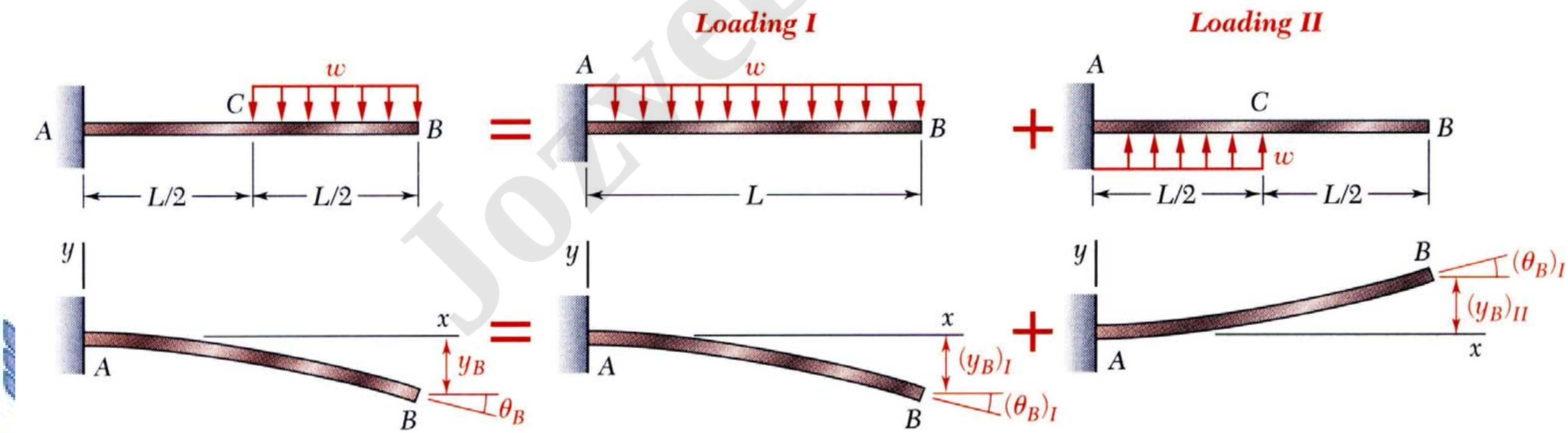
Sample Problem 9.7



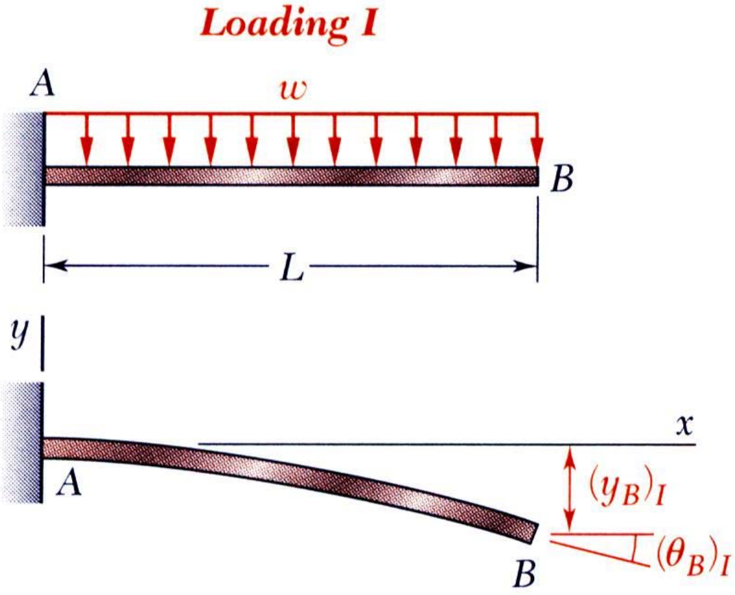
For the beam and loading shown, determine the slope and deflection at point *B*.

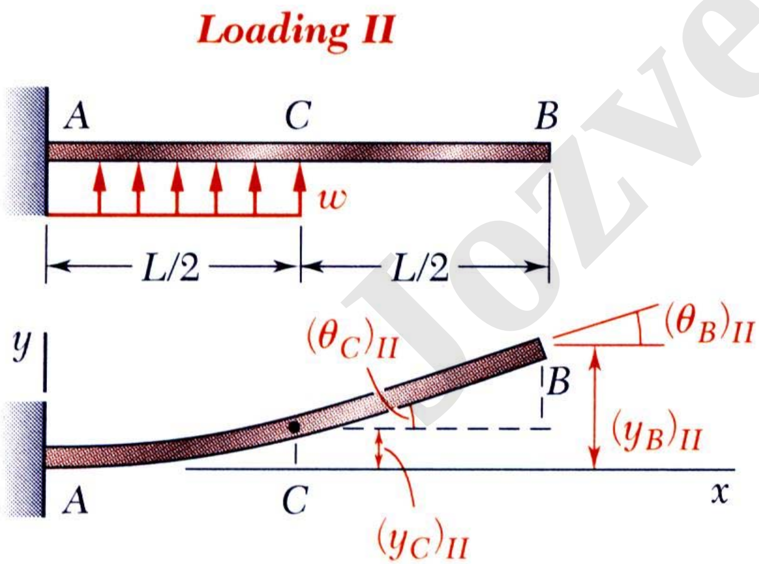
SOLUTION:

Superpose the deformations due to *Loading I* and *Loading II* as shown.



Sample Problem 9.7





Loading I

$$(\theta_B)_I = -\frac{wL^3}{6EI} \qquad (y_B)_I = -\frac{wL^4}{8EI}$$

Loading II

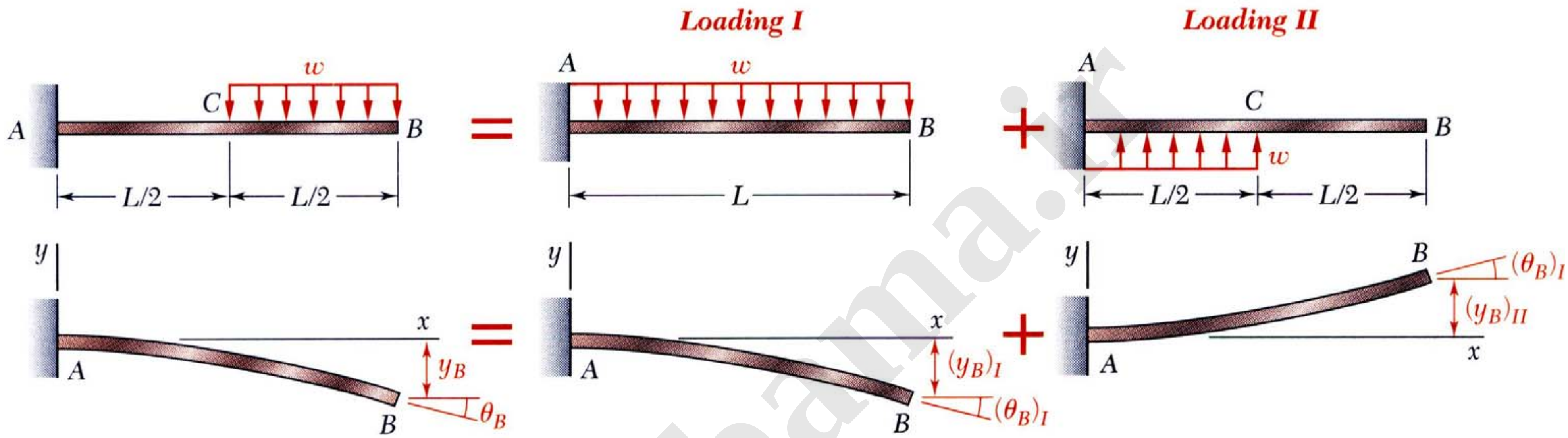
$$(\theta_C)_{II} = \frac{wL^3}{48EI} \qquad (y_C)_{II} = \frac{wL^4}{128EI}$$

In beam segment CB, the bending moment is zero and the elastic curve is a straight line.

$$(\theta_B)_{II} = (\theta_C)_{II} = \frac{wL^3}{48EI}$$

$$(y_B)_{II} = \frac{wL^4}{128EI} + \frac{wL^3}{48EI}\left(\frac{L}{2}\right) = \frac{7wL^4}{384EI}$$

Sample Problem 9.7



Combine the two solutions,

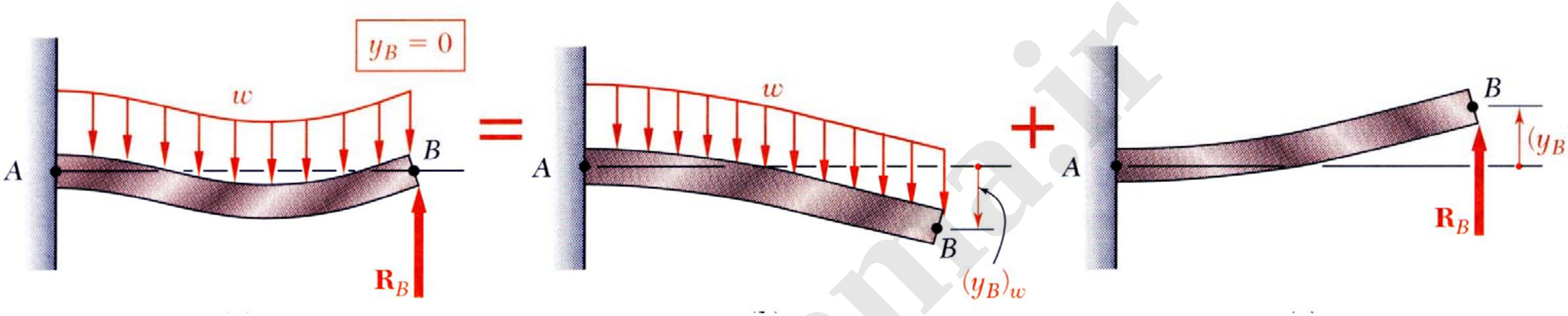
$$\theta_B = (\theta_B)_I + (\theta_B)_{II} = -\frac{wL^3}{6EI} + \frac{wL^3}{48EI}$$

$$\theta_B = \frac{7wL^3}{48EI}$$

$$y_B = (y_B)_I + (y_B)_{II} = -\frac{wL^4}{8EI} + \frac{7wL^4}{384EI}$$

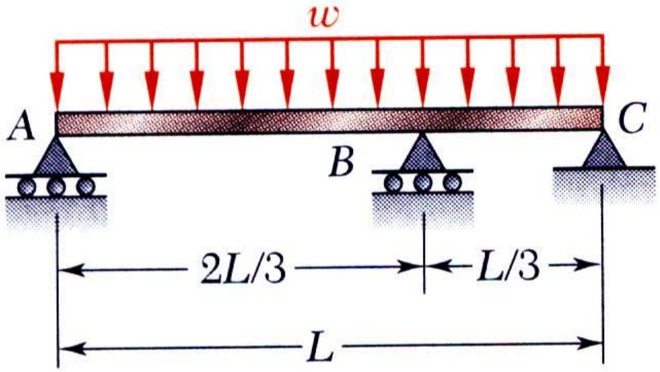
$$y_B = \frac{41wL^4}{384EI}$$

Application of Superposition to Statically Indeterminate Beams



- Method of superposition may be applied to determine the reactions at the supports of statically indeterminate beams.
- Designate one of the reactions as redundant and eliminate or modify the support.
- Determine the beam deformation without the redundant support.
- Treat the redundant reaction as an unknown load which, together with the other loads, must produce deformations compatible with the original supports.

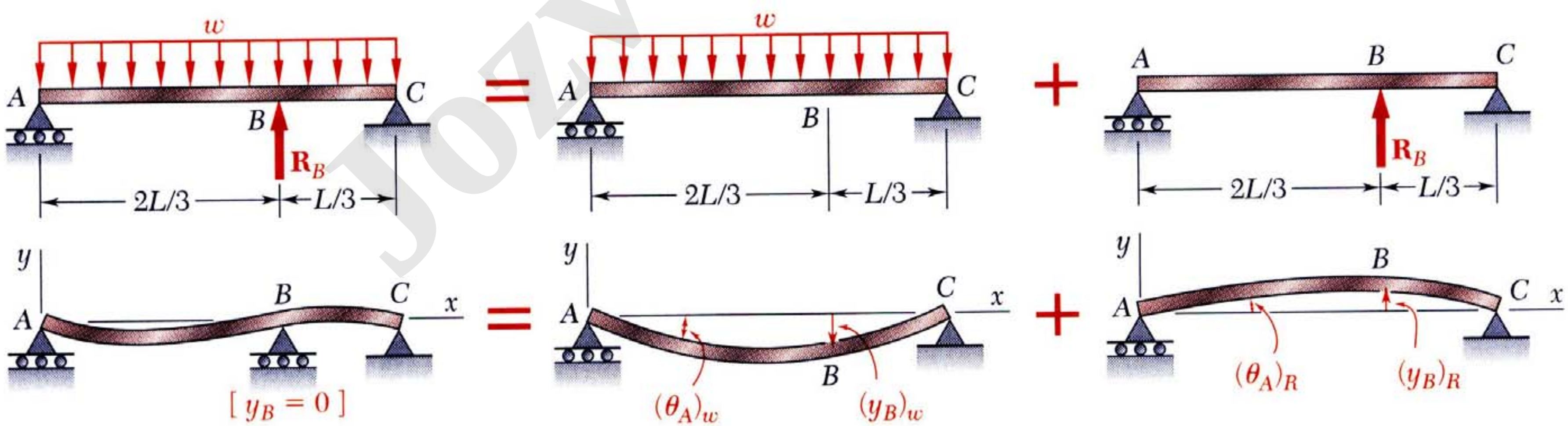
Sample Problem 9.8



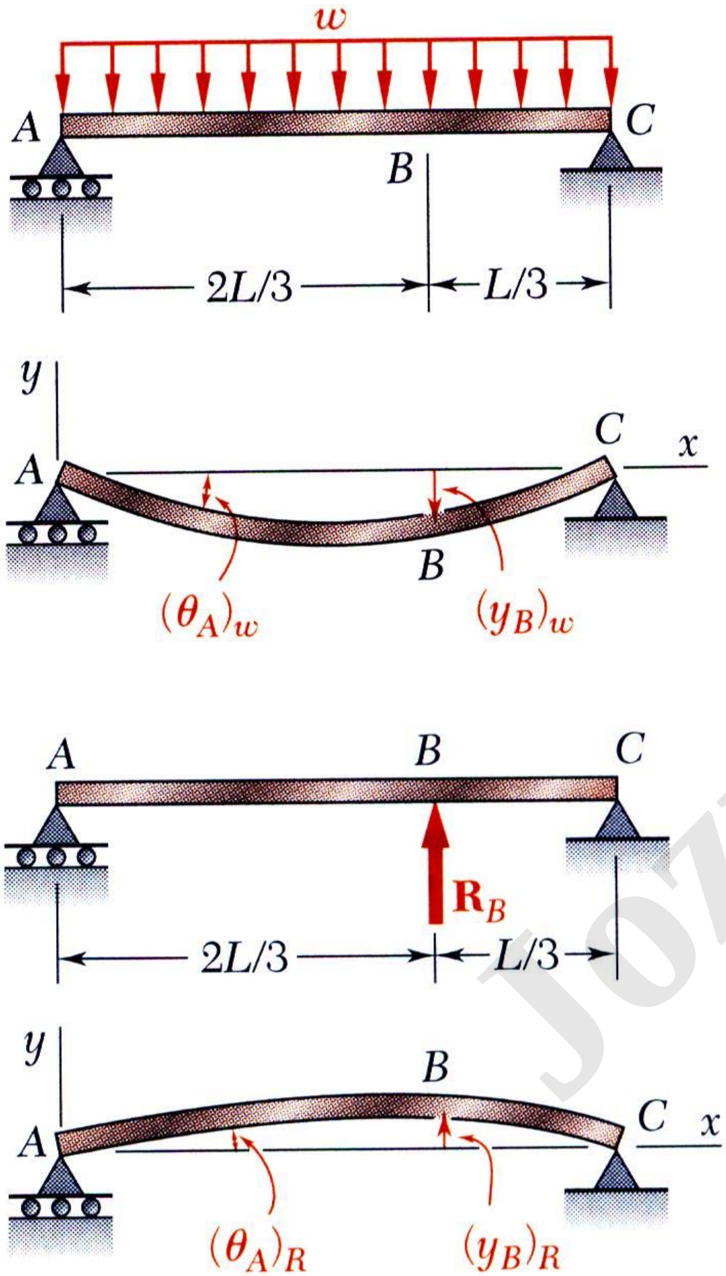
For the uniform beam and loading shown, determine the reaction at each support and the slope at end *A*.

SOLUTION:

- Release the "redundant" support at B, and find deformation.
- Apply reaction at *B* as an unknown load to force zero displacement at *B*.



Sample Problem 9.8



- Distributed Loading:

$$(y_B)_w = -\frac{w}{24EI}\left[\left(\frac{2}{3}L\right)^4 - 2L\left(\frac{2}{3}L\right)^3 + L^3\left(\frac{2}{3}L\right)\right]$$

$$= -0.01132\frac{wL^4}{EI}$$

- Redundant Reaction Loading:

$$(y_B)_R = \frac{R_B}{3EIL}\left(\frac{2}{3}L\right)^2\left(\frac{L}{3}\right)^2 = 0.01646\frac{R_B L^3}{EI}$$

- For compatibility with original supports, $y_B =$ 0

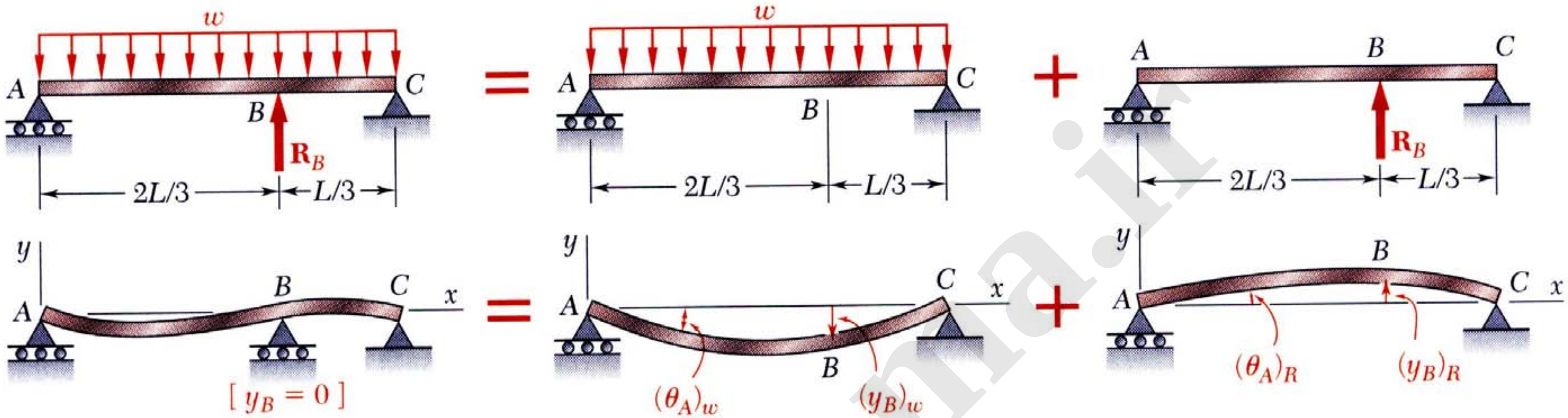
$$0 = (y_B)_w + (y_B)_R = -0.01132\frac{wL^4}{EI} + 0.01646\frac{R_B L^3}{EI}$$

$$R_B = 0.688wL \uparrow$$

- From statics,

$$R_A = 0.271wL \uparrow \qquad R_C = 0.0413wL \uparrow$$

Sample Problem 9.8



Slope at end *A*,

$$(\theta_A)_w = -\frac{wL^3}{24EI} = -0.04167\frac{wL^3}{EI}$$

$$(\theta_A)_R = \frac{0.0688wL}{6EIL}\left(\frac{L}{3}\right)\left[L^2 - \left(\frac{L}{3}\right)^2\right] = 0.03398\frac{wL^3}{EI}$$

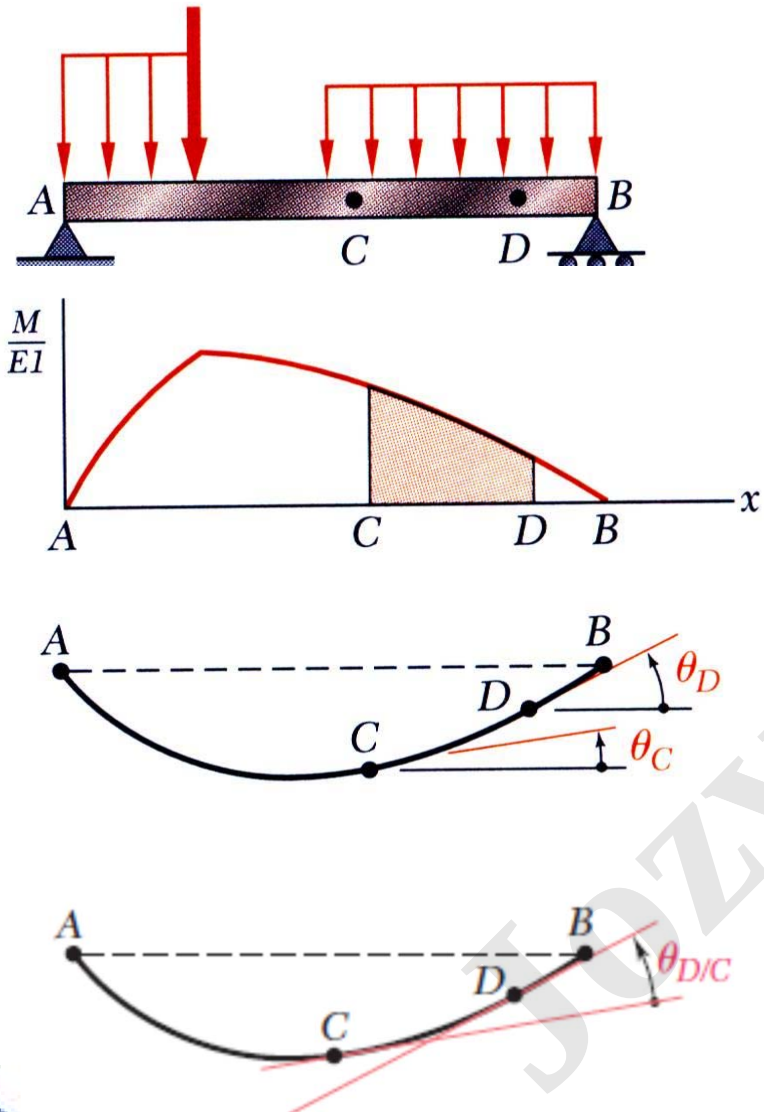
$$\theta_A = (\theta_A)_w + (\theta_A)_R = -0.04167\frac{wL^3}{EI} + 0.03398\frac{wL^3}{EI}$$

$$\theta_A = -0.00769\frac{wL^3}{EI}$$

تغییر شکل تیر

روش گشتاور سطح

Moment-Area Theorems



- Geometric properties of the elastic curve can be used to determine deflection and slope.

- Consider a beam subjected to arbitrary loading,

$$\frac{d\theta}{dx} = \frac{d^2y}{dx^2} = \frac{M}{EI} \qquad d\theta = \frac{M}{EI}\,dx$$

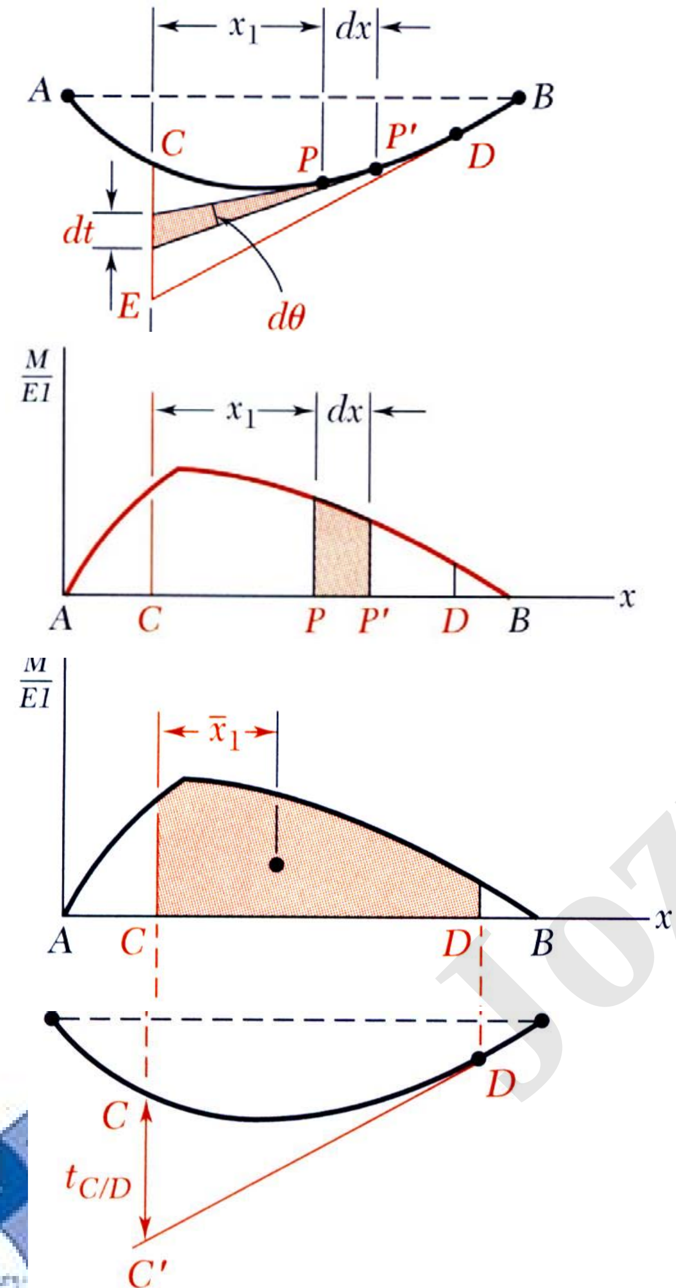
$$\int_{\theta_C}^{\theta_D} d\theta = \int_{x_C}^{x_D} \frac{M}{EI}\,dx$$

$$\theta_D - \theta_C = \int_{x_C}^{x_D} \frac{M}{EI}\,dx$$

- *First Moment-Area Theorem*:

  $\theta_{D/C} =$ area under *(M/EI)* diagram between *C* and *D*.

Moment-Area Theorems

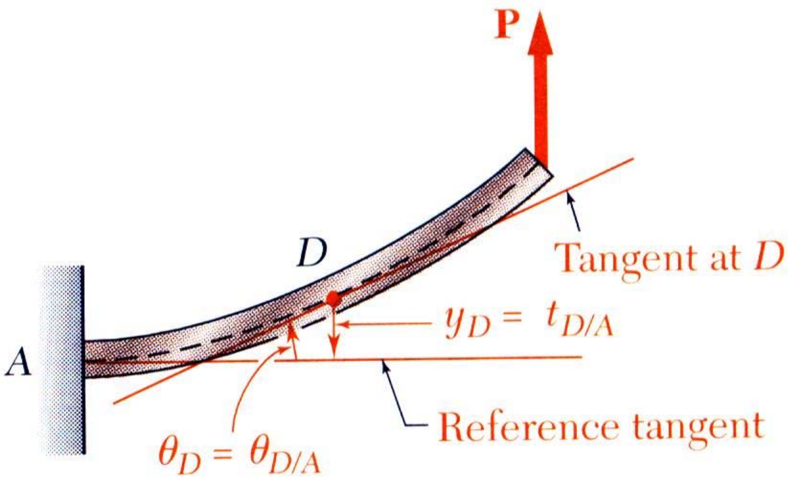


- Tangents to the elastic curve at *P* and *P'* intercept a segment of length *dt* on the vertical through *C*.

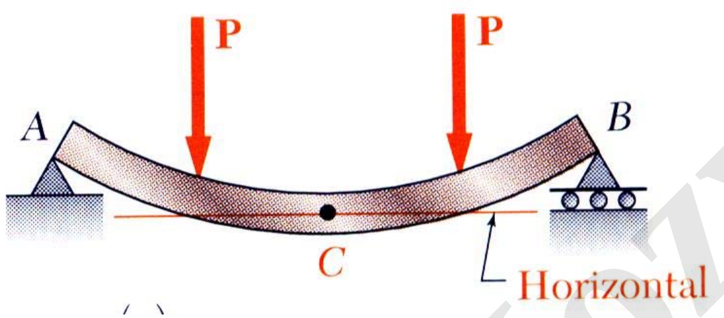
$$t_{C/D} = \int_{x_C}^{x_D} x_1 \frac{M}{EI} dx$$ = tangential deviation of *C* with respect to *D*

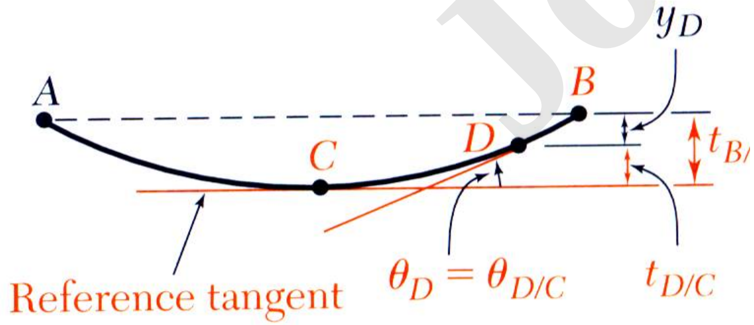
- *Second Moment-Area Theorem*:
  The tangential deviation of *C* with respect to *D* is equal to the first moment with respect to a vertical axis through *C* of the area under the (*M/EI*) diagram between *C* and *D*.

Application to Cantilever Beams and Beams With Symmetric Loadings

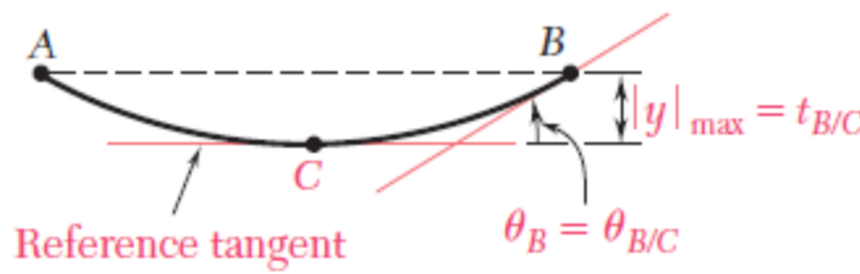


- Cantilever beam - Select tangent at A as the reference.



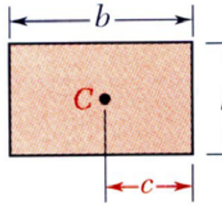
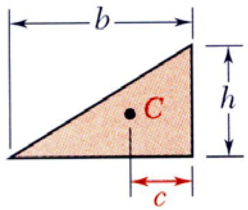
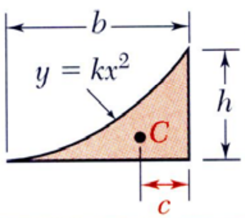
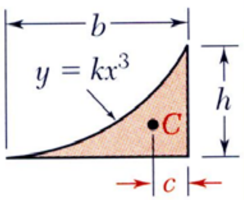
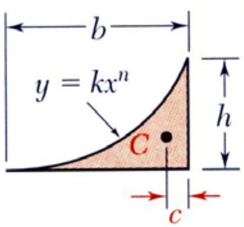


- Simply supported, symmetrically loaded beam - select tangent at C as the reference.



$$y_D = t_{D/C} - t_{B/C}.$$

Bending Moment Diagrams by Parts

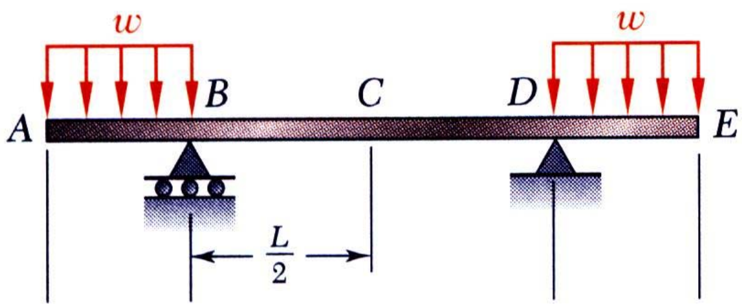
Shape		Area	c
Rectangle		bh	$\frac{b}{2}$
Triangle		$\frac{bh}{2}$	$\frac{b}{3}$
Parabolic spandrel		$\frac{bh}{3}$	$\frac{b}{4}$
Cubic spandrel		$\frac{bh}{4}$	$\frac{b}{5}$
General spandrel		$\frac{bh}{n+1}$	$\frac{b}{n+2}$

- Determination of the change of slope and the tangential deviation is simplified if the effect of each load is evaluated separately.

- Construct a separate (*M/EI*) diagram for each load.
  - The change of slope, $\theta_{D/C}$, is obtained by adding the areas under the diagrams.
  - The tangential deviation, $t_{D/C}$ is obtained by adding the first moments of the areas with respect to a vertical axis through D.

- Bending moment diagram constructed from individual loads is said to be *drawn by parts.*

Sample Problem 9.11

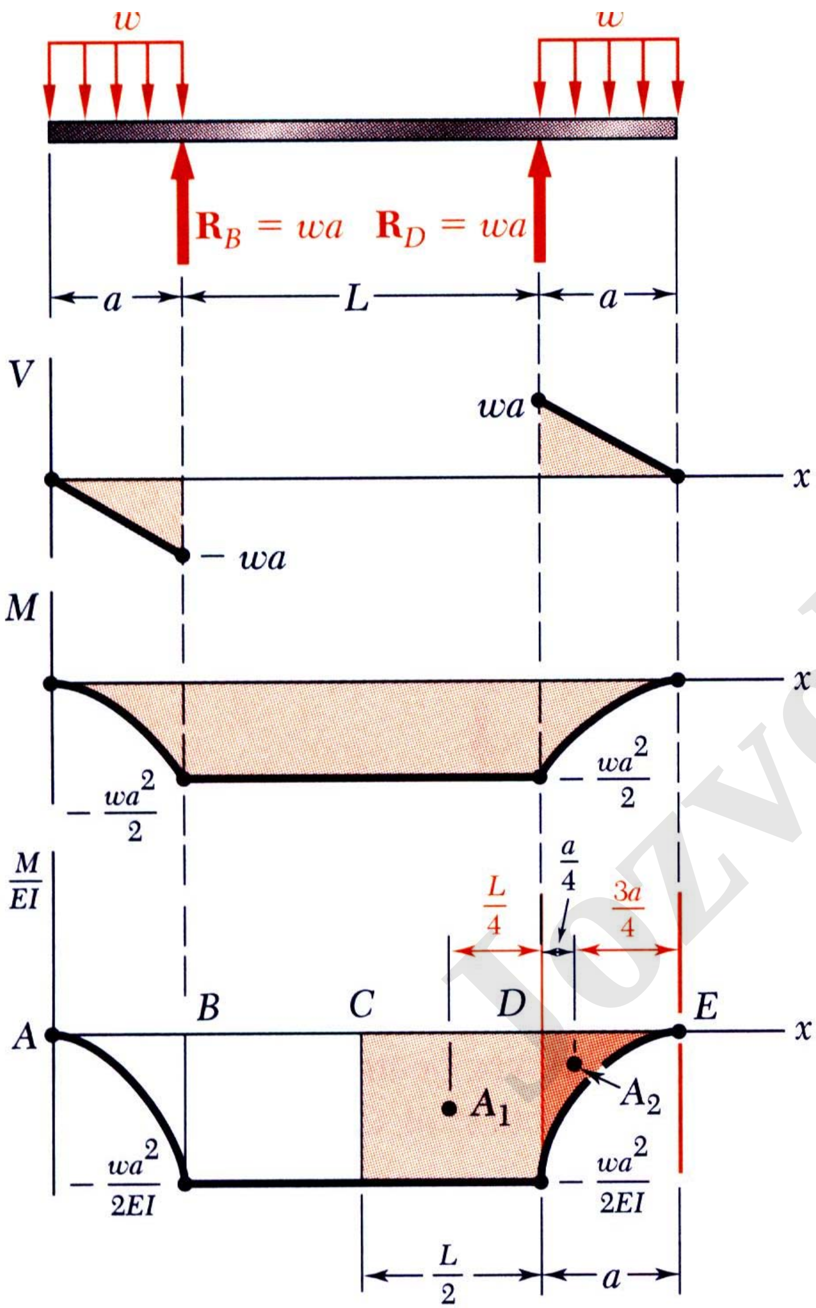


For the prismatic beam shown, determine the slope and deflection at *E*.

SOLUTION:

- Determine the reactions at supports.
- Construct shear, bending moment and (*M/EI*) diagrams.
- Taking the tangent at *C* as the reference, evaluate the slope and tangential deviations at *E*.

Sample Problem 9.11



SOLUTION:

- Determine the reactions at supports.

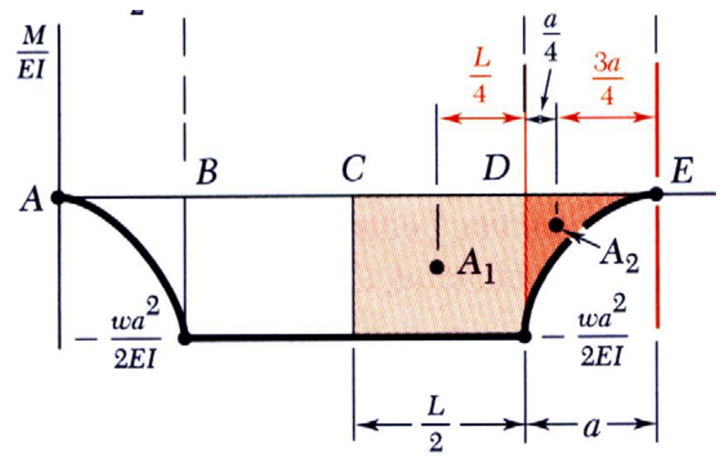
$$R_B = R_D = wa$$

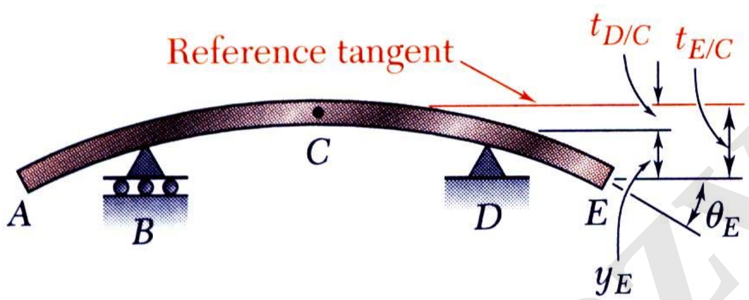
- Construct shear, bending moment and (*M/EI*) diagrams.

$$A_1 = -\frac{wa^2}{2EI}\left(\frac{L}{2}\right) = -\frac{wa^2 L}{4EI}$$

$$A_2 = -\frac{1}{3}\left(\frac{wa^2}{2EI}\right)(a) = -\frac{wa^3}{6EI}$$

Sample Problem 9.11





- Slope at E:

$$\theta_E = \theta_C + \theta_{E/C} = \theta_{E/C}$$

$$= A_1 + A_2 = -\frac{wa^2 L}{4EI} - \frac{wa^3}{6EI}$$

$$\theta_E = -\frac{wa^2}{12EI}(3L + 2a)$$

- Deflection at E:

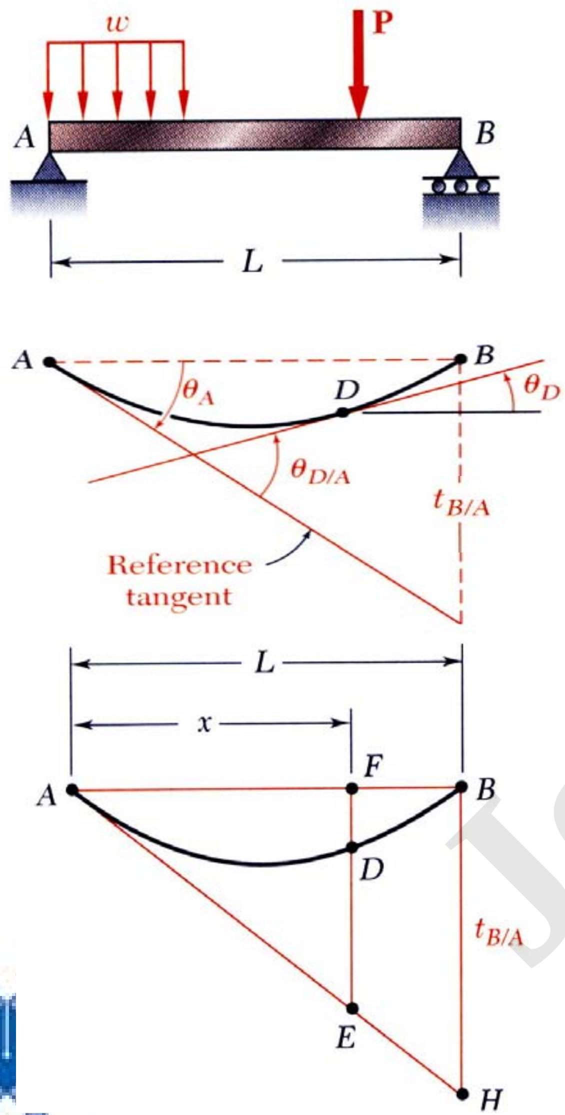
$$y_E = t_{E/C} - t_{D/C}$$

$$= \left[A_1\left(a + \frac{L}{4}\right) + A_2\left(\frac{3a}{4}\right) \right] - \left[A_1\left(\frac{L}{4}\right) \right]$$

$$= \left[-\frac{wa^3 L}{4EI} - \frac{wa^2 L^2}{16EI} - \frac{wa^4}{8EI} \right] - \left[-\frac{wa^2 L^2}{16EI} \right]$$

$$y_E = -\frac{wa^3}{8EI}(2L + a)$$

Application of Moment-Area Theorems to Beams With Unsymmetric Loadings



- Define reference tangent at support A. Evaluate θ_A by determining the tangential deviation at B with respect to A.

$$\theta_A = -\frac{t_{B/A}}{L}$$

- The slope at other points is found with respect to reference tangent.

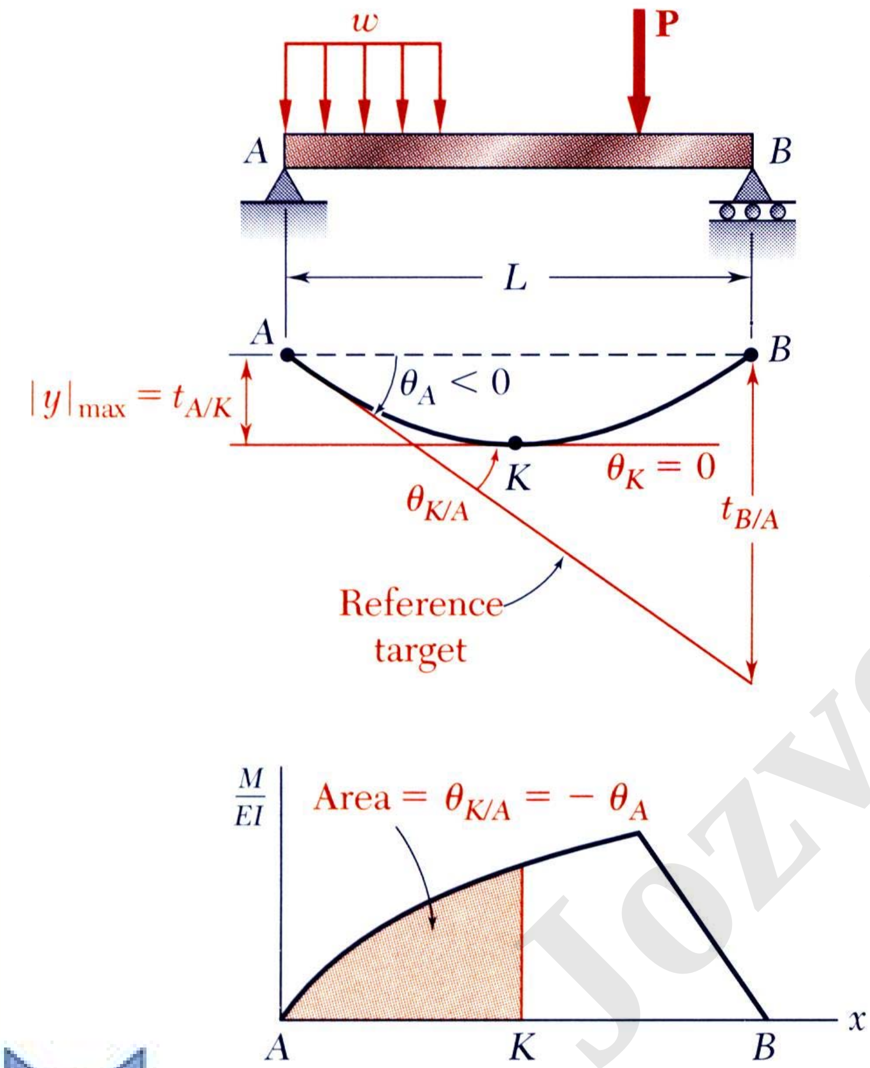
$$\theta_D = \theta_A + \theta_{D/A}$$

- The deflection at D is found from the tangential deviation at D.

$$\frac{EF}{x} = \frac{HB}{L} \quad \text{or} \quad EF = \frac{x}{L} t_{B/A}$$

$$y_D = ED - EF = t_{D/A} - \frac{x}{L} t_{B/A}$$

Maximum Deflection



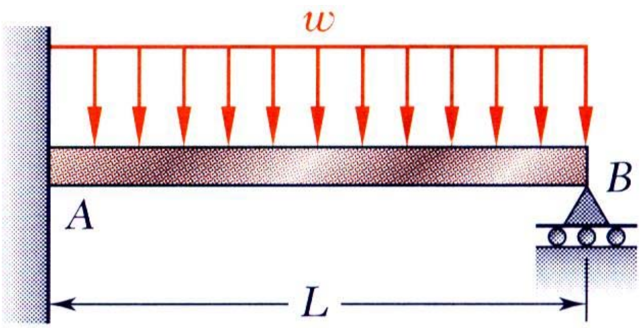
- Maximum deflection occurs at point *K* where the tangent is horizontal.

$$\theta_{K/A} = \theta_K - \theta_A = 0 - \theta_A = -\theta_A$$

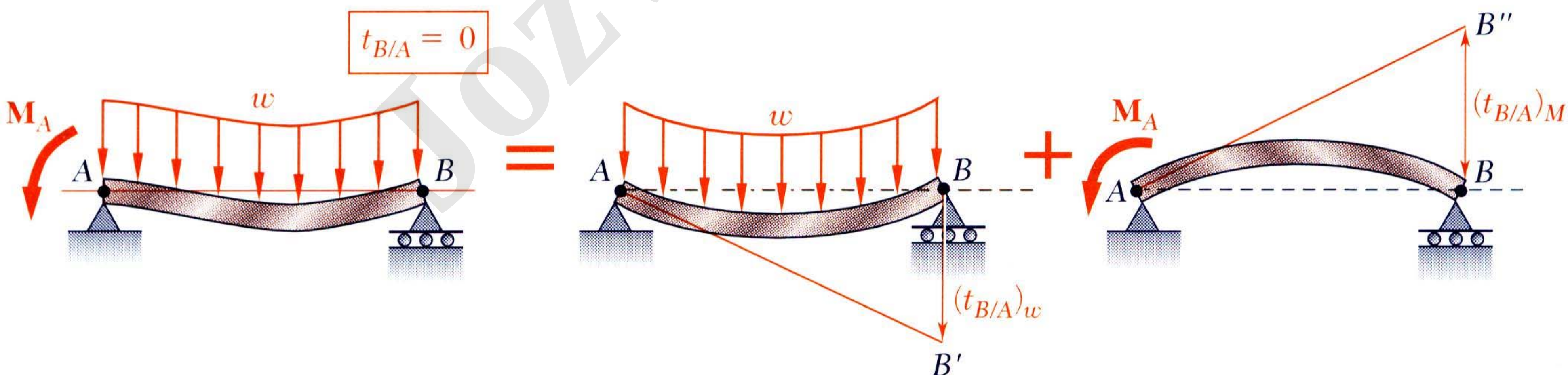
- Point *K* may be determined by measuring an area under the (*M/EI*) diagram equal to $-\theta_A$.

- Obtain y_{max} by computing the first moment with respect to the vertical axis through *A* of the area between *A* and *K*.

Use of Moment-Area Theorems With Statically Indeterminate Beams



- Reactions at supports of statically indeterminate beams are found by designating a redundant constraint and treating it as an unknown load which satisfies a displacement compatibility requirement.
- The (*M/EI*) diagram is drawn by parts. The resulting tangential deviations are superposed and related by the compatibility requirement.
- With reactions determined, the slope and deflection are found from the moment-area method.



تغییر شکل تیر

روش تیر مزدوج

6.6 Conjugate-Beam Method

The conjugate-beam method, developed by Otto Mohr in 1868, generally provides a more convenient means of computing slopes and deflections of beams than the moment-area method. Although the amount of computational effort required by the two methods is essentially the same, the conjugate-beam method is preferred by many engineers because of its systematic sign convention and straightforward application, which does not require sketching the elastic curve of the structure.

The conjugate-beam method is based on the analogy between the relationships among load, shear, and bending moment and the relationships among M/EI, slope, and deflection. These two types of relationships were derived in Sections 5.4 and 6.1, respectively, and are repeated in Table 6.1 for comparison purposes. As this table indicates, the relationships between M/EI, slope, and deflection have the same form as that of the relationships between load, shear, and bending moment. Therefore, the slope and deflection can be determined from M/EI by the same operations as those

TABLE 6.1

Load–Shear–Bending Moment Relationships	M/EI–Slope–Deflection Relationships
$\frac{dS}{dx} = w$	$\frac{d\theta}{dx} = \frac{M}{EI}$
$\frac{dM}{dx} = S$ or $\frac{d^2M}{dx^2} = w$	$\frac{dy}{dx} = \theta$ or $\frac{d^2y}{dx^2} = \frac{M}{EI}$

performed to compute shear and bending moment, respectively, from the load. Furthermore, if the M/EI diagram for a beam is applied as the load on a fictitious analogous beam, then the shear and bending moment at any point on the fictitious beam will be equal to the slope and deflection, respectively, at the corresponding point on the original real beam. The fictitious beam is referred to as the *conjugate beam*, and it is defined as follows:

> A conjugate beam corresponding to a real beam is a fictitious beam of the same length as the real beam, but it is externally supported and internally connected such that if the conjugate beam is loaded with the M/EI diagram of the real beam, the shear and bending moment at any point on the conjugate beam are equal, respectively, to the slope and deflection at the corresponding point on the real beam.

As the foregoing discussion indicates, the conjugate-beam method essentially involves computing the slopes and deflections of beams by computing the shears and bending moments in the corresponding conjugate beams.

Supports for Conjugate Beams

External supports and internal connections for conjugate beams are determined from the analogous relationships between conjugate beams and the corresponding real beams; that is, the shear and bending moment at any point on the conjugate beam must be consistent with the slope and deflection at that point on the real beam. The conjugate counterparts of the various types of real supports thus determined are shown in Fig. 6.12. As this figure indicates, a hinged or a roller support at an end of the real beam remains the same in the conjugate beam. This is because at such a support there may be slope, but no deflection, of the real beam. Therefore, at the corresponding end of the conjugate beam there must be shear but no bending moment; and a hinged or a roller support at that end would satisfy these conditions. Since at a fixed support of the real beam there is neither slope nor deflection, both shear and bending moment at that end of the conjugate beam must be zero; therefore, the conjugate of a fixed real support is a free end, as shown in Fig. 6.12. Conversely, a free end of a real beam becomes a fixed support in the conjugate beam because there may be slope as well as deflection at that end of the real beam; therefore, the conjugate beam must develop both shear and bending moment at that point. At an interior support of a real beam there is no deflection, but the slope is continuous (i.e., there is no abrupt change of slope from one side

Real Beam		Conjugate Beam	
Type of Support	Slope and Deflection	Shear and Bending Moment	Type of Support
Simple end support (or)	$\theta \neq 0$ $\Delta = 0$	$S \neq 0$ $M = 0$	Simple end support (or)
Fixed support	$\theta = 0$ $\Delta = 0$	$S = 0$ $M = 0$	Free end
Free end	$\theta \neq 0$ $\Delta \neq 0$	$S \neq 0$ $M \neq 0$	Fixed support
Simple interior support (or)	$\theta \neq 0$ and continuous $\Delta = 0$	$S \neq 0$ and continuous $M = 0$	Internal hinge
Internal hinge	$\theta \neq 0$ and discontinuous $\Delta \neq 0$	$S \neq 0$ and discontinuous $M \neq 0$	Simple interior support

FIG. 6.12 Supports for Conjugate Beams

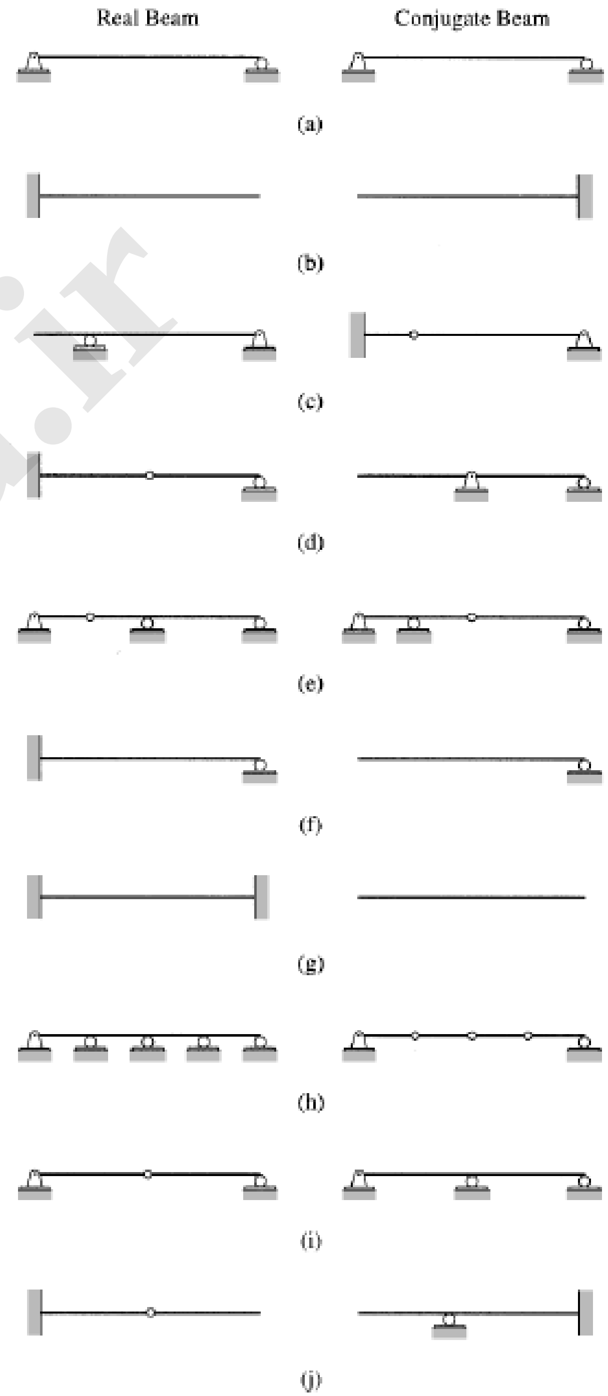


FIG. 6.13

Procedure for Analysis

The following step-by-step procedure can be used for determining the slopes and deflections of beams by the conjugate-beam method.

1. Construct the M/EI diagram for the given (real) beam subjected to the specified (real) loading. If the beam is subjected to a combination of different types of loads (e.g., concentrated loads and distributed loads), the analysis can be considerably expedited by constructing the M/EI diagram by parts, as discussed in the preceding section.
2. Determine the conjugate beam corresponding to the given real beam. The external supports and internal connections for the conjugate beam must be selected so that the shear and bending moment at any point on the conjugate beam are consistent with the slope and deflection, respectively, at that point on the real beam. The conjugates of various types of real supports are given in Fig. 6.12.
3. Apply the M/EI diagram (from step 1) as the load on the conjugate beam. The positive ordinates of the M/EI diagram are applied as upward loads on the conjugate beam and vice versa.
4. Calculate the reactions at the supports of the conjugate beam by applying the equations of equilibrium and condition (if any).
5. Determine the shears at those points on the conjugate beam where slopes are desired on the real beam. Determine the bending moments at those points on the conjugate beam where deflections are desired on the real beam. The shears and bending moments in conjugate beams are considered to be positive or negative in accordance with the beam sign convention (Fig. 5.2).
6. The slope at a point on the real beam with respect to the undeformed axis of the real beam is equal to the shear at that point on the conjugate beam. A positive shear in the conjugate beam denotes a positive or counterclockwise slope of the real beam and vice versa.
7. The deflection at a point on the real beam with respect to the undeformed axis of the real beam is equal to the bending moment at that point on the conjugate beam. A positive bending moment in the conjugate beam denotes a positive or upward deflection of the real beam and vice versa.

Example 6.8

Determine the slopes and deflections at points B and C of the cantilever beam shown in Fig. 6.14(a) by the conjugate-beam method.

Solution

***M/EI* Diagram.** This beam was analyzed in Example 6.3 by the moment-area method. The M/EI diagram for a reference moment of inertia $I = 3{,}000$ in.^{4} is shown in Fig. 6.14(b).

continued

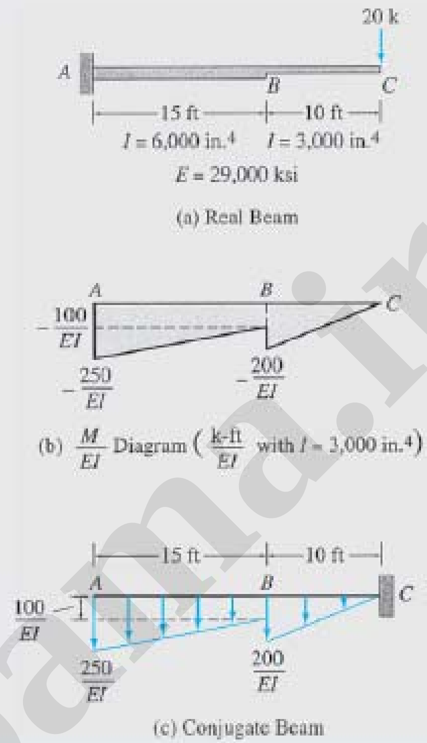


FIG. 6.14

Conjugate Beam. Fig. 6.14(c) shows the conjugate beam, loaded with the M/EI diagram of the real beam. Note that point A, which is fixed on the real beam, becomes free on the conjugate beam, whereas point C, which is free on the real beam, becomes fixed on the conjugate beam. Because the M/EI diagram is negative, it is applied as a downward load on the conjugate beam.

Slope at B. The slope at B on the real beam is equal to the shear at B in the conjugate beam. Using the free body of the conjugate beam to the left of B and considering the external forces acting upward on the free body as positive, in accordance with the beam sign convention (see Fig. 5.2), we compute the shear at B in the conjugate beam as

$$+\uparrow\ S_B = \frac{1}{EI}\left[-100(15) - \frac{1}{2}(150)(15)\right] = -\frac{2{,}625\ \text{k-ft}^2}{EI}$$

Therefore, the slope at B on the real beam is

$$\theta_B = -\frac{2{,}625\ \text{k-ft}^2}{EI}$$

Substituting the numerical values of E and I, we obtain

$$\theta_B = -\frac{2{,}625(12)^2}{(29{,}000)(3{,}000)} = -0.0043\ \text{rad}$$

$$\theta_B = 0.0043\ \text{rad}$$ **Ans.**

Deflection at B. The deflection at B on the real beam is equal to the bending moment at B in the conjugate beam. Using the free body of the conjugate beam to the left of B and considering the clockwise moments of the external forces about

continued

B as positive, in accordance with the beam sign convention (Fig. 5.2), we compute the bending moment at B on the conjugate beam as

$$+\circlearrowleft M_B = \frac{1}{EI}\left[-100(15)(7.5) - \frac{1}{2}(150)(15)(10)\right] = -\frac{22{,}500 \text{ k-ft}^3}{EI}$$

Therefore, the deflection at B on the real beam is

$$\Delta_B = -\frac{22{,}500 \text{ k-ft}^3}{EI} = -\frac{22{,}500(12)^3}{(29{,}000)(3{,}000)} = -0.45 \text{ in.}$$

$$\Delta_B = 0.45 \text{ in.} \downarrow$$ **Ans.**

Slope at C. Using the free body of the conjugate beam to the left of C, we determine the shear at C as

$$+\uparrow S_C = \frac{1}{EI}\left[-100(15) - \frac{1}{2}(150)(15) - \frac{1}{2}(200)(10)\right] = -\frac{3{,}625 \text{ k-ft}^2}{EI}$$

Therefore, the slope at C on the real beam is

$$\theta_C = -\frac{3{,}625 \text{ k-ft}^2}{EI} = -\frac{3{,}625(12)^2}{(29{,}000)(3{,}000)} = -0.006 \text{ rad}$$

$$\theta_C = 0.006 \text{ rad}$$ **Ans.**

Deflection at C. Considering the free body of the conjugate beam to the left of C, we obtain

$$+\circlearrowleft M_C = \frac{1}{EI}\left[-100(15)(17.5) - \frac{1}{2}(150)(15)(20) - \frac{1}{2}(200)(10)(6.67)\right]$$

$$= -\frac{55{,}420 \text{ k-ft}^3}{EI}$$

Therefore, the deflection at C on the real beam is

$$\Delta_C = -\frac{55{,}420 \text{ k-ft}^3}{EI} = -\frac{55{,}420(12)^3}{(29{,}000)(3{,}000)} = -1.1 \text{ in.}$$

$$\Delta_C = 1.1 \text{ in.} \downarrow$$ **Ans.**

Example 6.9

Determine the slope and deflection at point B of the beam shown in Fig. 6.15(a) by the conjugate-beam method.

Solution

***M/EI* Diagram.** See Fig. 6.15(b).

Conjugate Beam. The conjugate beam, loaded with the M/EI diagram of the real beam, is shown in Fig. 6.15(c).

Slope at B. Considering the free body of the conjugate beam to the left of B, we determine the shear at B as

$$+\uparrow S_B = \frac{M}{EI}(L) = \frac{ML}{EI}$$

continued

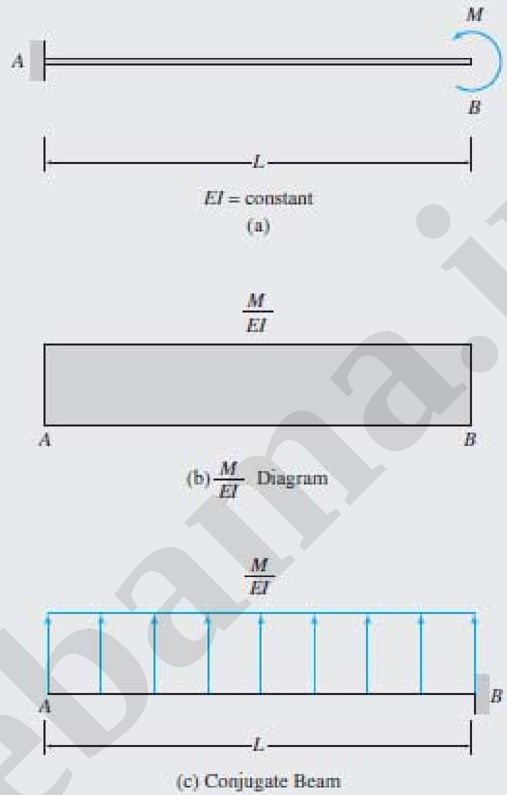


FIG. 6.15

Therefore, the slope at B on the real beam is

$$\theta_B = \frac{ML}{EI}$$

$$\theta_B = \frac{ML}{EI} \quad \text{(counterclockwise)} \qquad \textbf{Ans.}$$

Deflection at B. Using the free body of the conjugate beam to the left of B, we determine the bending moment at B as

$$+\circlearrowleft M_B = \frac{M}{EI}(L)\left(\frac{L}{2}\right) = \frac{ML^2}{2EI}$$

Therefore, the deflection at B on the real beam is

$$\Delta_B = \frac{ML^2}{2EI}$$

$$\Delta_B = \frac{ML^2}{2EI}\uparrow \qquad \textbf{Ans.}$$

Example 6.10

Use the conjugate-beam method to determine the slopes at ends *A* and *D* and the deflections at points *B* and *C* of the beam shown in Fig. 6.16(a).

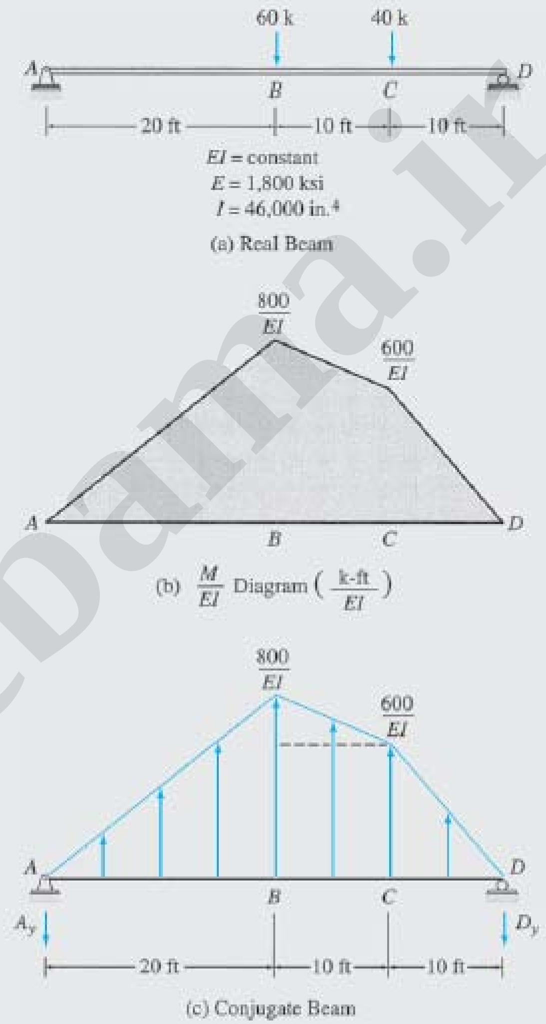


FIG. 6.16

Solution

M/EI Diagram. This beam was analyzed in Example 6.4 by the moment-area method. The M/EI diagram for this beam is shown in Fig. 6.16(b).

Conjugate Beam. Fig. 6.16(c) shows the conjugate beam loaded with the M/EI diagram of the real beam. Points *A* and *D*, which are simple end supports on the real beam, remain the same on the conjugate beam. Because the M/EI diagram is positive, it is applied as an upward load on the conjugate beam.

continued

Reactions for Conjugate Beam. By applying the equations of equilibrium to the free body of the entire conjugate beam, we obtain the following:

$$+\circlearrowleft \sum M_D = 0$$

$$A_y(40) - \frac{1}{EI}\left[\frac{1}{2}(800)(20)\left(\frac{20}{3}+20\right) + 600(10)(15) + \frac{1}{2}(200)(10)\left(\frac{20}{3}+10\right) + \frac{1}{2}(600)(10)\left(\frac{20}{3}\right)\right] = 0$$

$$A_y = \frac{8{,}500 \text{ k-ft}^2}{EI}$$

$$+\uparrow \sum F_y = 0$$

$$\frac{1}{EI}\left[-8{,}500 + \frac{1}{2}(800)(20) + 600(10) + \frac{1}{2}(200)(10) + \frac{1}{2}(600)(10)\right] - D_y = 0$$

$$D_y = \frac{9{,}500 \text{ k-ft}^2}{EI}$$

Slope at *A*. The slope at *A* on the real beam is equal to the shear just to the right of *A* in the conjugate beam, which is

$$+\uparrow S_{A,R} = -A_y = -\frac{8{,}500 \text{ k-ft}^2}{EI}$$

Therefore, the slope at *A* on the real beam is

$$\theta_A = -\frac{8{,}500 \text{ k-ft}^2}{EI} = -\frac{8{,}500(12)^2}{(1{,}800)(46{,}000)} = -0.015 \text{ rad}$$

$$\theta_A = 0.015 \text{ rad}$$ **Ans.**

Slope at *D*. The slope at *D* on the real beam is equal to the shear just to the left of *D* in the conjugate beam, which is

$$+\downarrow S_{D,L} = +D_y = \frac{+9{,}500 \text{ k-ft}^2}{EI}$$

Therefore, the slope at *D* on the real beam is

$$\theta_D = \frac{9{,}500 \text{ k-ft}^2}{EI} = \frac{9{,}500(12)^2}{(1{,}800)(46{,}000)} = 0.017 \text{ rad}$$

$$\theta_D = 0.017 \text{ rad}$$ **Ans.**

Deflection at *B*. The deflection at *B* on the real beam is equal to the bending moment at *B* in the conjugate beam. Using the free body of the conjugate beam to the left of *B*, we compute

$$+\circlearrowright M_B = \frac{1}{EI}\left[-8{,}500(20) + \frac{1}{2}(800)(20)\left(\frac{20}{3}\right)\right] = -\frac{116{,}666.67 \text{ k-ft}^3}{EI}$$

Therefore, the deflection at *B* on the real beam is

$$\Delta_B = -\frac{116{,}666.67 \text{ k-ft}^3}{EI} = -\frac{116{,}666.67(12)^3}{(1{,}800)(46{,}000)} = -2.43 \text{ in.}$$

$$\Delta_B = 2.43 \text{ in.} \downarrow$$ **Ans.**

continued

Deflection at C. The deflection at C on the real beam is equal to the bending moment at C in the conjugate beam. Using the free body of the conjugate beam to the right of C, we determine

$$+\circlearrowleft M_C = \frac{1}{EI}\left[-9{,}500(10) + \frac{1}{2}(600)(10)\left(\frac{10}{3}\right)\right] = -\frac{85{,}000 \text{ k-ft}^3}{EI}$$

Therefore, the deflection at C on the real beam is

$$\Delta_C = -\frac{85{,}000 \text{ k-ft}^3}{EI} = -\frac{85{,}000(12)^3}{(1{,}800)(46{,}000)} = -1.77 \text{ in.}$$

$$\Delta_C = 1.77 \text{ in.} \downarrow$$ **Ans.**

Example 6.11

Determine the maximum deflection for the beam shown in Fig. 6.17(a) by the conjugate-beam method.

Solution

***M/EI* Diagram.** This beam was previously analyzed in Example 6.5 by the moment-area method. The M/EI diagram for the beam is shown in Fig. 6.17(b).

Conjugate Beam. The simply supported conjugate beam, loaded with the M/EI diagram of the real beam, is shown in Fig. 6.17(c).

Reaction at Support *A* of the Conjugate Beam. By applying the moment equilibrium equation $\sum M_C = 0$ to the free body of the entire conjugate beam, we determine

$$+\circlearrowleft M_C = 0$$

$$A_y(15) - \frac{1}{EI}\left[\frac{1}{2}(400)(10)\left(\frac{10}{3}+5\right) + \frac{1}{2}(400)(5)\left(\frac{10}{3}\right)\right] = 0$$

$$A_y = \frac{1{,}333.33 \text{ kN}\cdot\text{m}^2}{EI}$$

Location of the Maximum Bending Moment in Conjugate Beam. If the maximum bending moment in the conjugate beam (or the maximum deflection on the real beam) occurs at point D, located at a distance x_m from the left support A (see Fig. 6.17(c)), then the shear in the conjugate beam at D must be zero. Considering the free body of the conjugate beam to the left of D, we write

$$+\uparrow S_D = \frac{1}{EI}\left[-1{,}333.33 + \frac{1}{2}(40x_m)(x_m)\right] = 0$$

from which

$$x_m = 8.16 \text{ m}$$

Maximum Deflection of the Real Beam. The maximum deflection of the real beam is equal to the maximum bending moment in the conjugate beam, which can be determined by considering the free body of the conjugate beam to the left of D, with $x_m = 8.16$ m. Thus,

$$+\circlearrowleft M_{max} = M_D = \frac{1}{EI}\left[-1{,}333.33(8.16) + \frac{1}{2}(40)(8.16)^2\left(\frac{8.16}{3}\right)\right]$$

$$= -\frac{7{,}244.51 \text{ kN}\cdot\text{m}^3}{EI}$$

continued

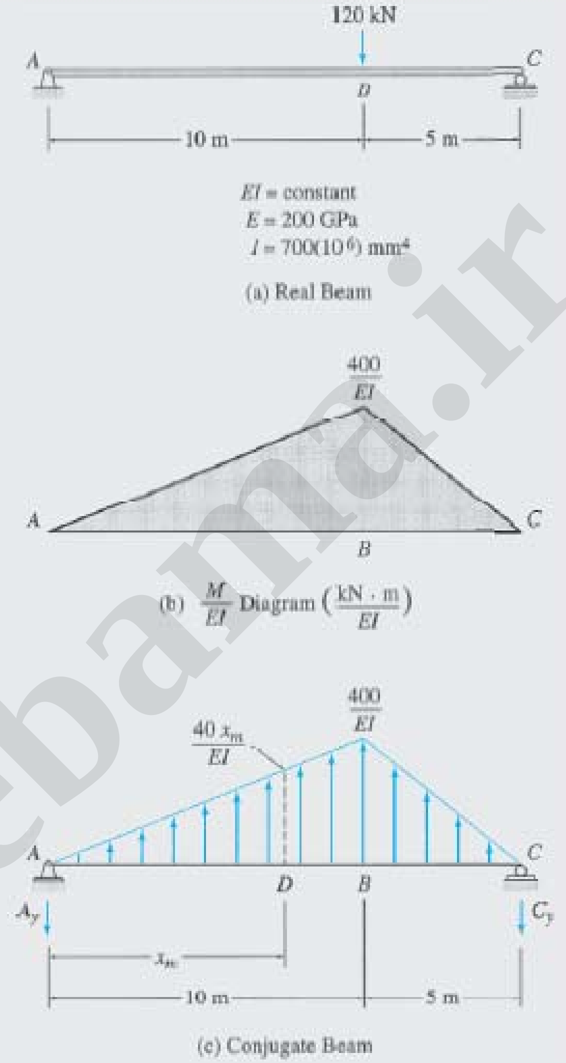


FIG. 6.17

Therefore, the maximum deflection of the real beam is

$$\Delta_{max} = -\frac{7{,}244.51 \text{ kN}\cdot\text{m}^3}{EI} = -\frac{7{,}244.51}{(200)(700)} = -0.0517 \text{ m} = -51.7 \text{ mm}$$

$$\Delta_{max} = 51.7 \text{ mm} \downarrow$$ **Ans.**

Example 6.12

Determine the slope at point A and the deflection at point C of the beam shown in Fig. 6.18(a) by the conjugate-beam method.

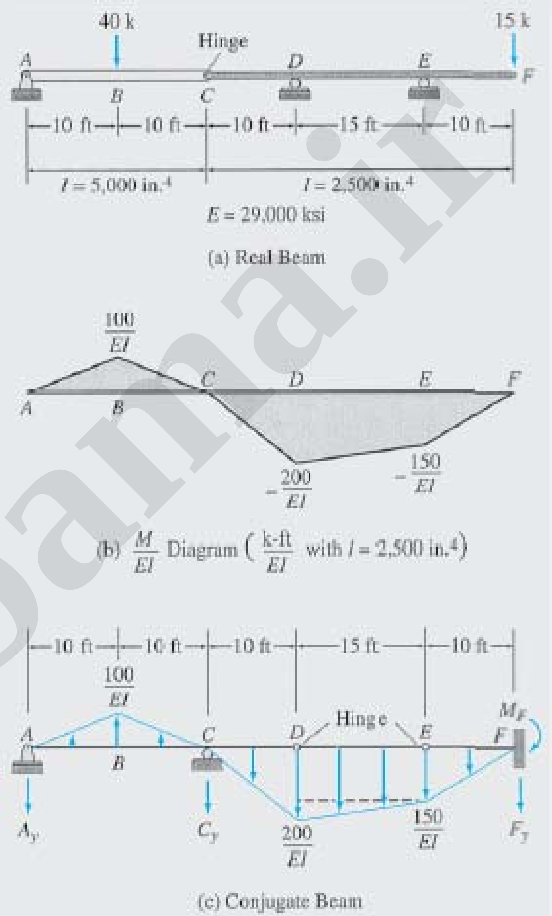


FIG. 6.18

Solution

M/EI Diagram. This beam was analyzed in Example 6.6 by the moment-area method. The M/EI diagram for a reference moment of inertia $I = 2{,}500$ in.^{4} is shown in Fig. 6.18(b).

Conjugate Beam. Figure 6.18(c) shows the conjugate beam loaded with the M/EI diagram of the real beam. Note that points D and E, which are simple interior supports on the real beam, become internal hinges on the conjugate beam; point C, which is an internal hinge on the real beam, becomes a simple interior support on the conjugate beam. Also note that the positive part of the M/EI diagram is applied as upward loading on the conjugate beam, whereas the negative part of the M/EI diagram is applied as downward loading.

continued

Reaction at Support *A* of the Conjugate Beam. We determine the reaction A_y of the conjugate beam by applying the equations of condition as follows:

$$+\circlearrowleft \sum M_D^{AD} = 0$$

$$A_y(30) - \frac{1}{2}\left(\frac{100}{EI}\right)(20)(20) + C_y(10) + \frac{1}{2}\left(\frac{200}{EI}\right)(10)\left(\frac{10}{3}\right) = 0$$

or

$$C_y = -3A_y + \frac{1{,}666.67}{EI} \tag{1}$$

$$+\circlearrowleft \sum M_E^{AE} = 0$$

$$A_y(45) - \frac{1}{2}\left(\frac{100}{EI}\right)(20)(35) + C_y(25) + \frac{1}{2}\left(\frac{200}{EI}\right)(10)\left(\frac{10}{3} + 15\right)$$

$$+ \frac{150}{EI}(15)(7.5) + \frac{1}{2}\left(\frac{50}{EI}\right)(15)(10) = 0$$

or

$$45A_y + 25C_y = -\frac{3{,}958.33}{EI} \tag{2}$$

Substituting Eq. (1) into Eq. (2) and solving for A_y, we obtain

$$A_y = \frac{1{,}520.83 \text{ k-ft}^2}{EI}$$

Slope at *A*. The slope at A on the real beam is equal to the shear just to the right of A in the conjugate beam, which is

$$+\uparrow S_{A,R} = -A_y = -\frac{1{,}520.83 \text{ k-ft}^2}{EI}$$

Therefore, the slope at A on the real beam is

$$\theta_A = -\frac{1{,}520.83}{EI} = -\frac{1{,}520.83(12)^2}{(29{,}000)(2{,}500)} = -0.003 \text{ rad}$$

$$\theta_A = 0.003 \text{ rad}$$ **Ans.**

Deflection at *C*. The deflection at C on the real beam is equal to the bending moment at C in the conjugate beam. Considering the free body of the conjugate beam to the left of C, we obtain

$$+\circlearrowleft M_C = \frac{1}{EI}\left[-1{,}520.83(20) + \frac{1}{2}(100)(20)(10)\right] = -\frac{20{,}416.67 \text{ k-ft}^3}{EI}$$

Therefore, the deflection at C on the real beam is

$$\Delta_C = -\frac{20{,}416.67 \text{ k-ft}^3}{EI} = -\frac{20{,}416.67(12)^3}{(29{,}000)(2{,}500)} = -0.487 \text{ in.}$$

$$\Delta_C = 0.487 \text{ in.} \downarrow$$ **Ans.**

Example 6.13

Use the conjugate-beam method to determine the deflection at point C of the beam shown in Fig. 6.19(a).

Solution

***M/EI* Diagram.** This beam was previously analyzed in Example 6.7 by the moment-area method. The M/EI diagram by cantilever parts with respect to point B is shown in Fig. 6.19(b).

continued

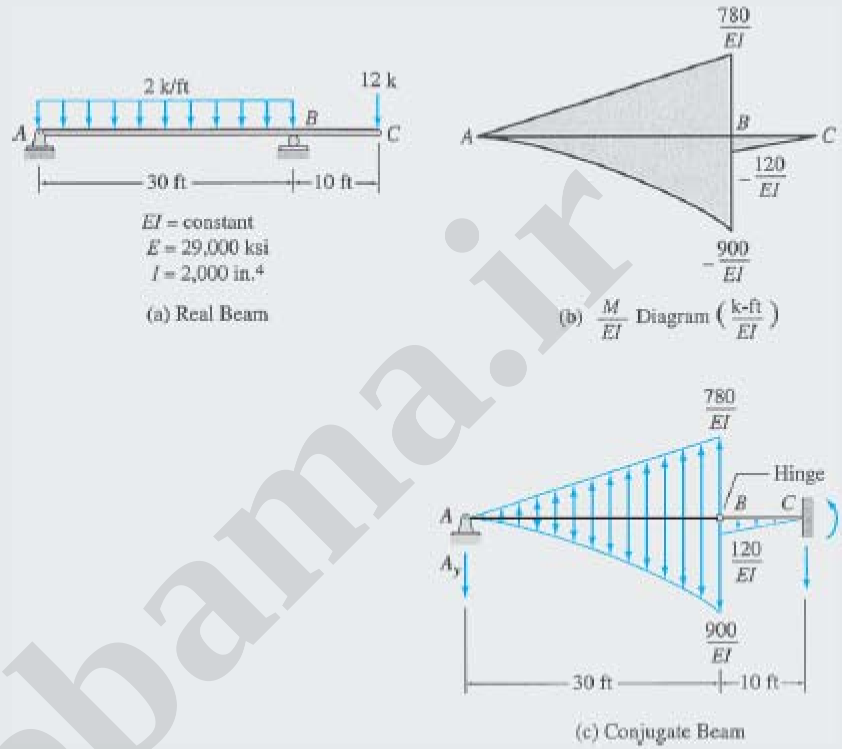


FIG. 6.19

Conjugate Beam. See Fig. 6.19(c).

Reaction at Support A **of the Conjugate Beam.**

$$+\circlearrowleft \sum M_B^{AB} = 0$$

$$A_y(30) + \frac{1}{EI}\left[\frac{1}{3}(900)(30)\left(\frac{30}{4}\right) - \frac{1}{2}(780)(30)\left(\frac{30}{3}\right)\right] = 0$$

$$A_y = \frac{1{,}650 \text{ k-ft}^2}{EI}$$

Deflection at C**.** The deflection at C on the real beam is equal to the bending moment at C in the conjugate beam. Considering the free body of the conjugate beam to the left of C, we obtain

$$+\circlearrowleft M_C = \frac{1}{EI}\left[-1{,}650(40) - \frac{1}{3}(900)(30)\left(\frac{30}{4} + 10\right) + \frac{1}{2}(780)(30)(20)\right.$$
$$\left. - \frac{1}{2}(120)(10)\left(\frac{20}{3}\right)\right] = \frac{6{,}500 \text{ k-ft}^3}{EI}$$

Therefore, the deflection at C on the real beam is

$$\Delta_C = \frac{6{,}500 \text{ k-ft}^3}{EI} = \frac{6{,}500(12)^3}{(29{,}000)(2{,}000)} = 0.194 \text{ in.}$$

$$\Delta_C = 0.194 \text{ in.} \uparrow$$ **Ans.**

Summary

In this chapter we have discussed the geometric methods for determining the slopes and deflections of statically determinate beams. The differential equation for the deflection of beams can be expressed as

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \tag{6.9}$$

The direct integration method essentially involves writing expression(s) for M/EI for the beam in terms of x and integrating the expression(s) successively to obtain equations for the slope and deflection of the elastic curve. The constants of integration are determined from the boundary conditions and the conditions of continuity of the elastic curve. If a beam is subjected to several loads, the slope or deflection due to the combined effects of the loads can be determined by algebraically adding the slopes or deflections due to each of the loads acting individually on the beam.

The moment-area method is based on two theorems, which can be mathematically expressed as follows:

$$\text{First moment-area theorem:} \quad \theta_{BA} = \int_A^B \frac{M}{EI}\,dx \tag{6.12}$$

$$\text{Second moment-area theorem:} \quad \Delta_{BA} = \int_A^B \frac{M}{EI}\bar{x}\,dx \tag{6.15}$$

Two procedures for constructing bending moment diagrams by parts are presented in Section 6.5.

A conjugate beam is a fictitious beam of the same length as the corresponding real beam; but it is externally supported and internally connected such that, if the conjugate beam is loaded with the M/EI diagram of the real beam, the shear and bending moment at any point on the conjugate beam are equal, respectively, to the slope and deflection at the corresponding point on the real beam. The conjugate-beam method essentially involves determining the slopes and deflections of beams by computing the shears and bending moments in the corresponding conjugate beams.

PROBLEMS

Section 6.2

6.1 through 6.6 Determine the equations for slope and deflection of the beam shown by the direct integration method. EI = constant.

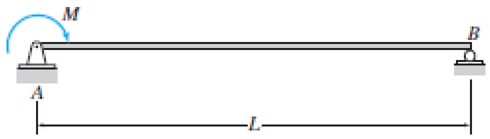


FIG. P6.1

Example 6.1

Determine the equations for the slope and deflection of the beam shown in Fig. 6.2(a) by the direct integration method. Also, compute the slope at each end and the deflection at the midspan of the beam. *EI* is constant.

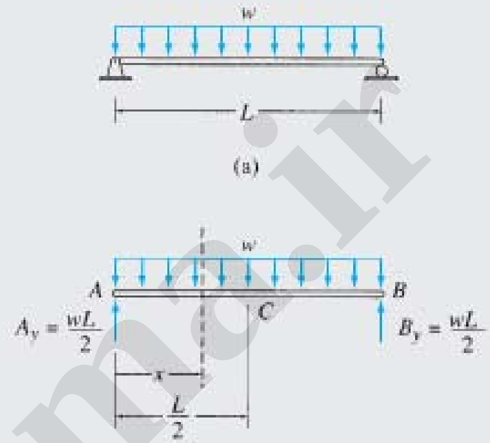


FIG. 6.2

Solution

Reactions. See Fig. 6.2(b).

$$+\rightarrow \sum F_x = 0 \qquad A_x = 0$$

$$+\circlearrowleft \sum M_B = 0$$

$$-A_y(L) + w(L)\left(\frac{L}{2}\right) = 0 \qquad A_y = \frac{wL}{2}\uparrow$$

$$+\uparrow \sum F_y = 0$$

$$\left(\frac{wL}{2}\right) - (wL) + B_y = 0 \qquad B_y = \frac{wL}{2}\uparrow$$

Equation for Bending Moment. To determine the equation for bending moment for the beam, we pass a section at a distance x from support A, as shown in Fig. 6.2(b). Considering the free body to the left of this section, we obtain

$$M = \frac{wL}{2}(x) - (wx)\left(\frac{x}{2}\right) = \frac{w}{2}(Lx - x^2)$$

Equation for *M*/*EI*. The flexural rigidity, *EI*, of the beam is constant, so the equation for M/EI can be written as

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{w}{2EI}(Lx - x^2)$$

continued

Equations for Slope and Deflection. The equation for the slope of the elastic curve of the beam can be obtained by integrating the equation for M/EI as

$$\theta = \frac{dy}{dx} = \frac{w}{2EI}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_1$$

Integrating once more, we obtain the equation for deflection as

$$y = \frac{w}{2EI}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) + C_1 x + C_2$$

The constants of integration, C_1 and C_2, are evaluated by applying the following boundary conditions:

At end A, $\quad x = 0, \quad y = 0$

At end B, $\quad x = L, \quad y = 0$

By applying the first boundary condition—that is, by setting $x = 0$ and $y = 0$ in the equation for y—we obtain $C_2 = 0$. Next, by using the second boundary condition—that is, by setting $x = L$ and $y = 0$ in the equation for y—we obtain

$$0 = \frac{w}{2EI}\left(\frac{L^4}{6} - \frac{L^4}{12}\right) + C_1 L$$

from which

$$C_1 = -\frac{wL^3}{24EI}$$

Thus, the equations for slope and deflection of the beam are

$$\theta = \frac{w}{2EI}\left(\frac{Lx^2}{2} - \frac{x^3}{3} - \frac{L^3}{12}\right) \qquad (1) \qquad \textbf{Ans.}$$

$$y = \frac{wx}{12EI}\left(Lx^2 - \frac{x^3}{2} - \frac{L^3}{2}\right) \qquad (2) \qquad \textbf{Ans.}$$

Slopes at Ends A and B. By substituting $x = 0$ and L, respectively, into Eq. (1), we obtain

$$\theta_A = -\frac{wL^3}{24EI} \quad \text{or} \quad \theta_A = \frac{wL^3}{24EI} \qquad \textbf{Ans.}$$

$$\theta_B = \frac{wL^3}{24EI} \quad \text{or} \quad \theta_B = \frac{wL^3}{24EI} \qquad \textbf{Ans.}$$

Deflection at Midspan. By substituting $x = L/2$ into Eq. (2), we obtain

$$y_C = -\frac{5wL^4}{384EI} \quad \text{or} \quad y_C = \frac{5wL^4}{384EI}\downarrow \qquad \textbf{Ans.}$$

Example 6.2

Determine the slope and deflection at point B of the cantilever beam shown in Fig. 6.3(a) by the direct integration method.

Solution

Equation for Bending Moment. We pass a section at a distance x from support A, as shown in Fig. 6.3(b). Considering the free body to the right of this section, we write the equation for bending moment as

$$M = -15(20 - x)$$

continued

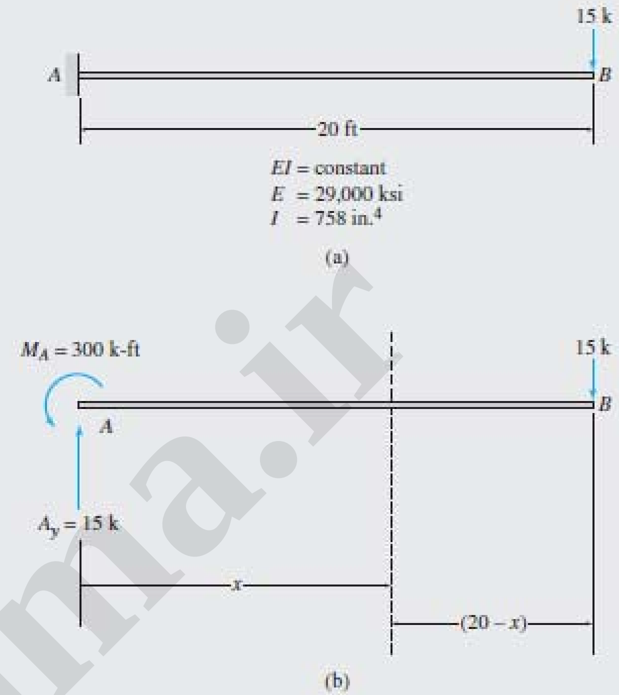


FIG. 6.3

Equation for ***M/EI.***

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{15}{EI}(20 - x)$$

Equations for Slope and Deflection. By integrating the equation for M/EI, we determine the equation for slope as

$$\theta = \frac{dy}{dx} = -\frac{15}{EI}\left(20x - \frac{x^2}{2}\right) + C_1$$

Integrating once more, we obtain the equation for deflection as

$$y = -\frac{15}{EI}\left(10x^2 - \frac{x^3}{6}\right) + C_1x + C_2$$

The constants of integration, C_1 and C_2, are evaluated by using the boundary conditions that $\theta = 0$ at $x = 0$, and $y = 0$ at $x = 0$. By applying the first boundary condition—that is, by setting $\theta = 0$ and $x = 0$ in the equation for θ—we obtain $C_1 = 0$. Similarly, by applying the second boundary condition—that is, by setting $y = 0$ and $x = 0$ in the equation for y—we obtain $C_2 = 0$. Thus, the equations for slope and deflection of the beam are

$$\theta = -\frac{15}{EI}\left(20x - \frac{x^2}{2}\right)$$

$$y = -\frac{15}{EI}\left(10x^2 - \frac{x^3}{6}\right)$$

Slope and Deflection at End ***B.*** By substituting $x = 20$ ft, $E = 29{,}000(12^2)$ ksf, and $I = 758/(12^4)$ ft^4 into the foregoing equations for slope and deflection, we obtain

$\theta_B = -0.0197$ rad or $\theta_B = 0.0197$ rad ↻ **Ans.**

$y_B = -0.262$ ft $= -3.14$ in. or $y_B = 3.14$ in. ↓ **Ans.**

Example 6.3

Determine the slopes and deflections at points *B* and *C* of the cantilever beam shown in Fig. 6.5(a) by the moment-area method.

Solution

Bending Moment Diagram. The bending moment diagram for the beam is shown in Fig. 6.5(b).

continued

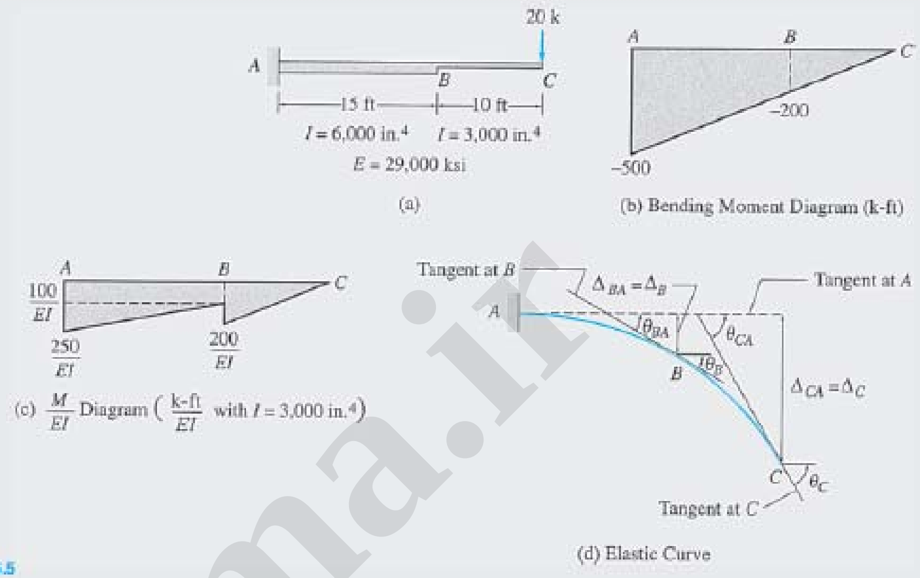


FIG. 6.5

***M/EI* Diagram.** As indicated in Fig. 6.5(a), the values of the moment of inertia of the segments AB and BC of the beam are 6,000 in.^{4} and 3,000 in.^{4}, respectively. Using $I = I_{BC} = 3{,}000$ in.^{4} as the reference moment of inertia, we express I_{AB} in terms of I as

$$I_{AB} = 6{,}000 = 2(3{,}000) = 2I$$

which indicates that in order to obtain the M/EI diagram in terms of EI, we must divide the bending moment diagram for segment AB by 2, as shown in Fig. 6.5(c).

Elastic Curve. The elastic curve for the beam is shown in Fig. 6.5(d). Note that because the M/EI diagram is negative, the beam bends concave downward. Since the support at A is fixed, the slope at A is zero ($\theta_A = 0$); that is, the tangent to the elastic curve at A is horizontal, as shown in the figure.

Slope at *B*. With the slope at A known, we can determine the slope at B by evaluating the change in slope θ_{BA} between A and B (which is the angle between the tangents to the elastic curve at points A and B, as shown in Fig. 6.5(d)). According to the first moment-area theorem, θ_{BA} = area of the M/EI diagram between A and B. This area can be conveniently evaluated by dividing the M/EI diagram into triangular and rectangular parts, as shown in Fig. 6.5(c). Thus,

$$\theta_{BA} = \frac{1}{EI}\left[(100)(15) + \frac{1}{2}(150)(15)\right] = \frac{2{,}625 \text{ k-ft}^2}{EI}$$

From Fig. 6.5(d), we can see that because the tangent at A is horizontal (in the direction of the undeformed axis of the beam), the slope at $B(\theta_B)$ is equal to the angle θ_{BA} between the tangents at A and B; that is,

$$\theta_B = \theta_{BA} = \frac{2{,}625 \text{ k-ft}^2}{EI} = \frac{2{,}625(12)^2 \text{ k-in.}^2}{EI}$$

Substituting the numerical values of $E = 29{,}000$ ksi and $I = 3{,}000$ in.^{4}, we obtain

$$\theta_B = \frac{2{,}625(12)^2}{(29{,}000)(3{,}000)} \text{ rad} = 0.0043 \text{ rad}$$

$$\theta_B = 0.0043 \text{ rad}$$ **Ans.**

continued

Deflection at *B*. From Fig. 6.5(d), it can be seen that the deflection of *B* with respect to the undeformed axis of the beam is equal to the tangential deviation of *B* from the tangent at *A*; that is,

$$\Delta_B = \Delta_{BA}$$

According to the second moment-area theorem,

$$\Delta_{BA} = \text{moment of the area of the } M/EI \text{ diagram between } A \text{ and } B \text{ about } B$$

$$= \frac{1}{EI}\left[(100)(15)(7.5) + \frac{1}{2}(150)(15)(10)\right] = \frac{22{,}500 \text{ k-ft}^3}{EI}$$

Therefore,

$$\Delta_B = \Delta_{BA} = \frac{22{,}500 \text{ k-ft}^3}{EI}$$

$$= \frac{22{,}500(12)^3}{(29{,}000)(3{,}000)} = 0.45 \text{ in.}$$

$$\Delta_B = 0.45 \text{ in.} \downarrow$$ **Ans.**

Slope at *C*. From Fig. 6.5(d), we can see that

$$\theta_C = \theta_{CA}$$

where

$$\theta_{CA} = \text{area of the } M/EI \text{ diagram between } A \text{ and } C$$

$$= \frac{1}{EI}\left[(100)(15) + \frac{1}{2}(150)(15) + \frac{1}{2}(200)(10)\right] = \frac{3{,}625 \text{ k-ft}^2}{EI}$$

Therefore,

$$\theta_C = \theta_{CA} = \frac{3{,}625 \text{ k-ft}^2}{EI}$$

$$= \frac{3{,}625(12)^2}{(29{,}000)(3{,}000)} = 0.006 \text{ rad}$$

$$\theta_C = 0.006 \text{ rad}$$ **Ans.**

Deflection at *C*. It can be seen from Fig. 6.5(d) that

$$\Delta_C = \Delta_{CA}$$

where

$$\Delta_{CA} = \text{moment of the area of the } M/EI \text{ diagram between } A \text{ and } C \text{ about } C$$

$$= \frac{1}{EI}\left[(100)(15)(7.5+10) + \frac{1}{2}(150)(15)(10+10) + \frac{1}{2}(200)(10)(6.67)\right]$$

$$= \frac{55{,}420 \text{ k-ft}^3}{EI}$$

Therefore,

$$\Delta_C = \Delta_{CA} = \frac{55{,}420 \text{ k-ft}^3}{EI}$$

$$= \frac{55{,}420(12)^3}{(29{,}000)(3{,}000)} = 1.1 \text{ in.}$$

$$\Delta_C = 1.1 \text{ in.} \downarrow$$ **Ans.**

Example 6.4

Use the moment-area method to determine the slopes at ends A and D and the deflections at points B and C of the beam shown in Fig. 6.6(a).

Solution

***M/EI* Diagram.** Because EI is constant along the length of the beam, the shape of the M/EI diagram is the same as that of the bending moment diagram. The M/EI diagram is shown in Fig. 6.6(b).

Elastic Curve. The elastic curve for the beam is shown in Fig. 6.6(c).

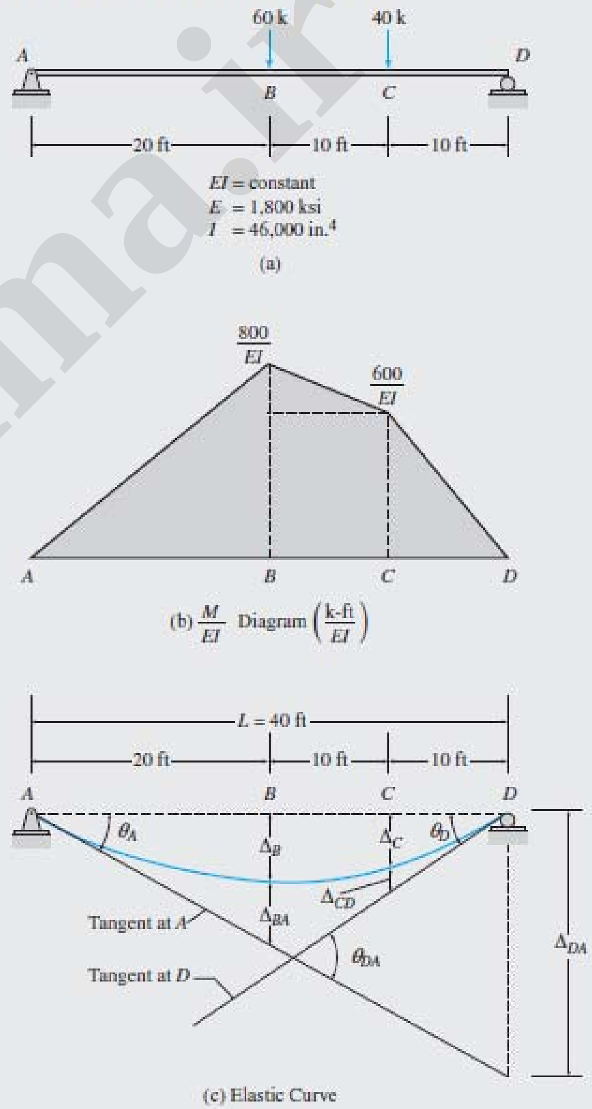


FIG. 6.6

continued

Slope at *A*. The slope of the elastic curve is not known at any point on the beam, so we will use the tangent at support A as the reference tangent and determine its slope, θ_A, from the conditions that the deflections at the support points A and D are zero. From Fig. 6.6(c), we can see that

$$\theta_A = \frac{\Delta_{DA}}{L}$$

in which θ_A is assumed to be so small that $\tan\theta_A \approx \theta_A$. To evaluate the tangential deviation Δ_{DA}, we apply the second moment-area theorem:

$$\Delta_{DA} = \text{moment of the area of the } M/EI \text{ diagram between } A \text{ and } D \text{ about } D$$

$$\Delta_{DA} = \frac{1}{EI}\left[\frac{1}{2}(800)(20)\left(\frac{20}{3}+20\right)+\frac{1}{2}(200)(10)\left(\frac{20}{3}+10\right) + 600(10)(15)+\frac{1}{2}(600)(10)\left(\frac{20}{3}\right)\right]$$

$$= \frac{340{,}000 \text{ k-ft}^3}{EI}$$

Therefore, the slope at A is

$$\theta_A = \frac{\Delta_{DA}}{L} = \frac{340{,}000/EI}{40} = \frac{8{,}500 \text{ k-ft}^2}{EI}$$

Substituting the numerical values of E and I, we obtain

$$\theta_A = \frac{8{,}500(12)^2}{(1{,}800)(46{,}000)} = 0.015 \text{ rad}$$

$$\theta_A = 0.015 \text{ rad}$$ **Ans.**

Slope at *D*. From Fig. 6.6(c), we can see that

$$\theta_D = \theta_{DA} - \theta_A$$

in which, according to the first moment-area theorem,

$$\theta_{DA} = \text{area of the } M/EI \text{ diagram between } A \text{ and } D$$

$$= \frac{1}{EI}\left[\frac{1}{2}(800)(20)+\frac{1}{2}(200)(10)+600(10)+\frac{1}{2}(600)(10)\right]$$

$$= \frac{18{,}000 \text{ k-ft}^2}{EI}$$

Therefore,

$$\theta_D = \frac{18{,}000}{EI} - \frac{8{,}500}{EI} = \frac{9{,}500 \text{ k-ft}^2}{EI}$$

$$\theta_D = \frac{9{,}500(12)^2}{(1{,}800)(46{,}000)} = 0.017 \text{ rad}$$

$$\theta_D = 0.017 \text{ rad}$$ **Ans.**

Deflection at *B*. Considering the portion AB of the elastic curve in Fig. 6.6(c), and realizing that θ_A is so small that $\tan\theta_A \approx \theta_A$, we write

$$\theta_A = \frac{\Delta_B + \Delta_{BA}}{20}$$

from which

$$\Delta_B = 20\theta_A - \Delta_{BA}$$

continued

where

$$\Delta_{BA} = \text{moment of the area of the } M/EI \text{ diagram between } A \text{ and } B \text{ about } B$$

$$= \frac{1}{EI}\left[\frac{1}{2}(800)(20)\left(\frac{20}{3}\right)\right]$$

$$= \frac{53{,}333.33 \text{ k-ft}^3}{EI}$$

Therefore,

$$\Delta_B = 20\left(\frac{8{,}500}{EI}\right) - \frac{53{,}333.33}{EI} = \frac{116{,}666.67 \text{ k-ft}^3}{EI}$$

$$\Delta_B = \frac{116{,}666.67(12)^3}{(1{,}800)(46{,}000)} = 2.43 \text{ in.}$$

$$\Delta_B = 2.43 \text{ in.} \downarrow$$ **Ans.**

Deflection at *C*. Finally, considering the portion *CD* of the elastic curve in Fig. 6.6(c) and assuming θ_D to be small (so that $\tan \theta_D \approx \theta_D$), we write

$$\theta_D = \frac{\Delta_C + \Delta_{CD}}{10}$$

or

$$\Delta_C = 10\theta_D - \Delta_{CD}$$

where

$$\Delta_{CD} = \frac{1}{EI}\left[\frac{1}{2}(600)(10)\left(\frac{10}{3}\right)\right] = \frac{10{,}000 \text{ k-ft}^3}{EI}$$

Therefore,

$$\Delta_C = 10\left(\frac{9{,}500}{EI}\right) - \frac{10{,}000}{EI} = \frac{85{,}000 \text{ k-ft}^3}{EI}$$

$$\Delta_C = \frac{85{,}000(12)^3}{(1{,}800)(46{,}000)} = 1.77 \text{ in.}$$

$$\Delta_C = 1.77 \text{ in.} \downarrow$$ **Ans.**

Example 6.5

Determine the maximum deflection for the beam shown in Fig. 6.7(a) by the moment-area method.

Solution

***M/EI* Diagram.** The *M/EI* diagram is shown in Fig. 6.7(b).

Elastic Curve. The elastic curve for the beam is shown in Fig. 6.7(c).

continued

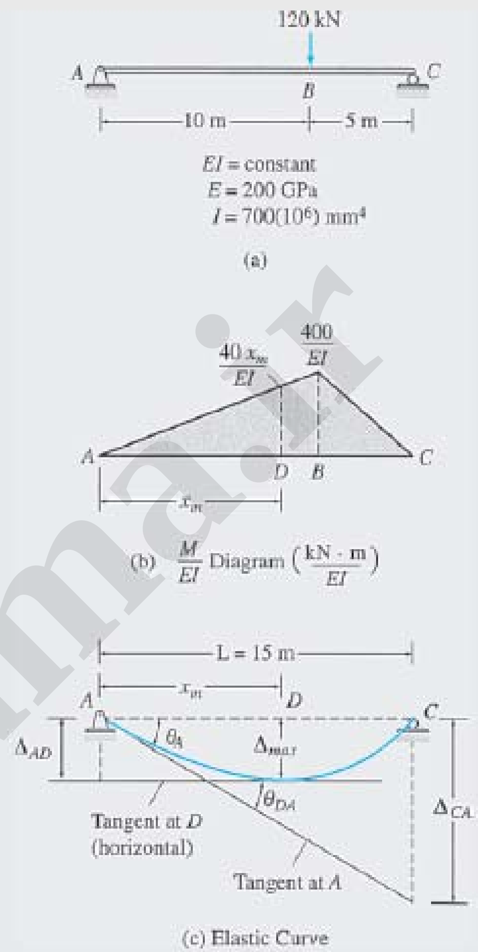


FIG. 6.7

Slope at ***A*****.** The slope of the elastic curve is not known at any point on the beam, so we will use the tangent at support A as the reference tangent and determine its slope, θ_A, from the conditions that the deflections at the support points A and C are zero. From Fig. 6.7(c), we can see that

$$\theta_A = \frac{\Delta_{CA}}{15}$$

To evaluate the tangential deviation Δ_{CA}, we apply the second moment-area theorem:

$$\Delta_{CA} = \text{moment of the area of the } M/EI \text{ diagram between } A \text{ and } C \text{ about } C$$

$$\Delta_{CA} = \frac{1}{EI}\left[\frac{1}{2}(400)(10)\left(\frac{10}{3}+5\right)+\frac{1}{2}(400)(5)\left(\frac{10}{3}\right)\right]$$

$$= \frac{20{,}000 \text{ kN} \cdot \text{m}^3}{EI}$$

continued

Therefore, the slope at A is

$$\theta_A = \frac{20{,}000/EI}{15} = \frac{1{,}333.33 \text{ kN}\cdot\text{m}^2}{EI}$$

Location of the Maximum Deflection. If the maximum deflection occurs at point D, located at a distance x_m from the left support A (see Fig. 6.7(c)), then the slope at D must be zero; therefore,

$$\theta_{DA} = \theta_A = \frac{1{,}333.33 \text{ kN}\cdot\text{m}^2}{EI}$$

which indicates that in order for the slope at D to be zero (i.e., the maximum deflection occurs at D), the area of the M/EI diagram between A and D must be equal to $1{,}333.33/EI$. We use this condition to determine the location of point D:

$$\theta_{DA} = \text{area of the } \frac{M}{EI} \text{ diagram between } A \text{ and } D = \frac{1{,}333.33}{EI}$$

or

$$\frac{1}{2}\left(\frac{40x_m}{EI}\right)x_m = \frac{1{,}333.33}{EI}$$

from which

$$x_m = 8.16 \text{ m}$$

Maximum Deflection. From Fig. 6.7(c), we can see that

$$\Delta_{max} = \Delta_{AD}$$

where

$$\Delta_{AD} = \text{moment of the area of the } M/EI \text{ diagram between } A \text{ and } D \text{ about } A$$

$$= \frac{1}{2}\frac{(40)(8.16)}{EI}(8.16)\left(\frac{2}{3}\right)(8.16)$$

$$= \frac{7{,}244.51 \text{ kN}\cdot\text{m}^3}{EI}$$

Therefore,

$$\Delta_{max} = \frac{7{,}244.51 \text{ kN}\cdot\text{m}^3}{EI}$$

Substituting $E = 200$ GPa $= 200(10^6)$ kN/m^2 and $I = 700(10^6)$ mm^4 $= 700(10^{-6})$ m^4, we obtain

$$\Delta_{max} = \frac{7{,}244.51}{200(10^6)(700)(10^{-6})} = 0.0517 \text{ m}$$

$$\Delta_{max} = 51.7 \text{ mm} \downarrow$$ **Ans.**

Example 6.6

Use the moment-area method to determine the slope at point A and the deflection at point C of the beam shown in Fig. 6.8(a).

Solution

***M/EI* Diagram.** The bending moment diagram is shown in Fig. 6.8(b), and the M/EI diagram for a reference moment of inertia $I = 2{,}500$ in.^{4} is shown in Fig. 6.8(c).

continued

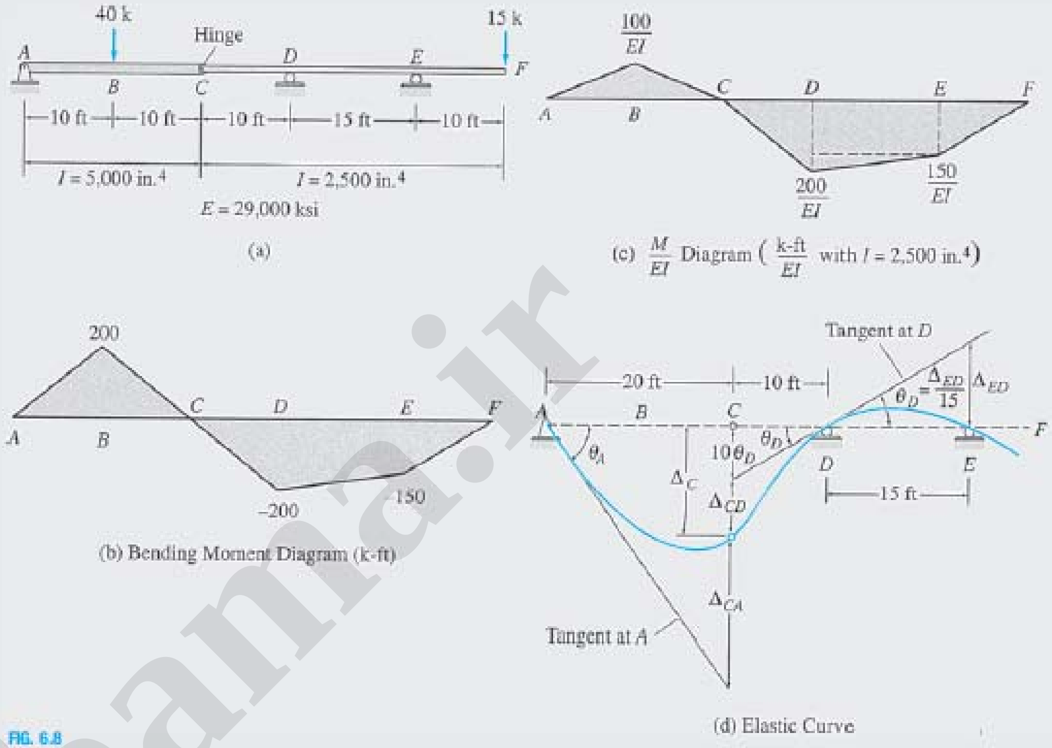


FIG. 6.8

Elastic Curve. The elastic curve for the beam is shown in Fig. 6.8(d). Note that the elastic curve is discontinuous at the internal hinge C. Therefore, the moment-area theorems must be applied separately over the portions AC and CF of the curve on each side of the hinge.

Slope at D. The tangent at support D is selected as the reference tangent. From Fig. 6.8(d), we can see that the slope of this tangent is given by the relationship

$$\theta_D = \frac{\Delta_{ED}}{15}$$

where, from the second moment-area theorem,

$$\Delta_{ED} = \frac{1}{EI}\left[150(15)(7.5) + \frac{1}{2}(50)(15)(10)\right] = \frac{20{,}625 \text{ k-ft}^3}{EI}$$

Therefore,

$$\theta_D = \frac{20{,}625}{15(EI)} = \frac{1{,}375 \text{ k-ft}^2}{EI}$$

Deflection at C. From Fig. 6.8(d), we can see that

$$\Delta_C = 10\theta_D + \Delta_{CD}$$

in which

$$\Delta_{CD} = \frac{1}{2}\left(\frac{200}{EI}\right)(10)\left(\frac{20}{3}\right) = \frac{6{,}666.67 \text{ k-ft}^3}{EI}$$

continued

Therefore,

$$\Delta_C = 10\left(\frac{1,375}{EI}\right) + \frac{6,666.67}{EI} = \frac{20,416.67 \text{ k-ft}^3}{EI}$$

Substituting the numerical values of E and I, we obtain

$$\Delta_C = \frac{20,416.67(12)^3}{(29,000)(2,500)} = 0.487 \text{ in.}$$

$$\Delta_C = 0.487 \text{ in.} \downarrow$$ **Ans.**

Slope at A. Considering the portion AC of the elastic curve, we can see from Fig. 6.8(d) that

$$\theta_A = \frac{\Delta_C + \Delta_{CA}}{20}$$

where

$$\Delta_{CA} = \frac{1}{2}\left(\frac{100}{EI}\right)(20)(10) = \frac{10,000 \text{ k-ft}^3}{EI}$$

Therefore,

$$\theta_A = \frac{1}{20}\left(\frac{20,416.67}{EI} + \frac{10,000}{EI}\right) = \frac{1,520.83 \text{ k-ft}^2}{EI}$$

$$\theta_A = \frac{1,520.83(12)^2}{(29,000)(2,500)} = 0.003 \text{ rad}$$

$$\theta_A = 0.003 \text{ rad}$$ **Ans.**

Example 6.7

Determine the deflection at point C of the beam shown in Fig. 6.11(a) by the moment-area method.

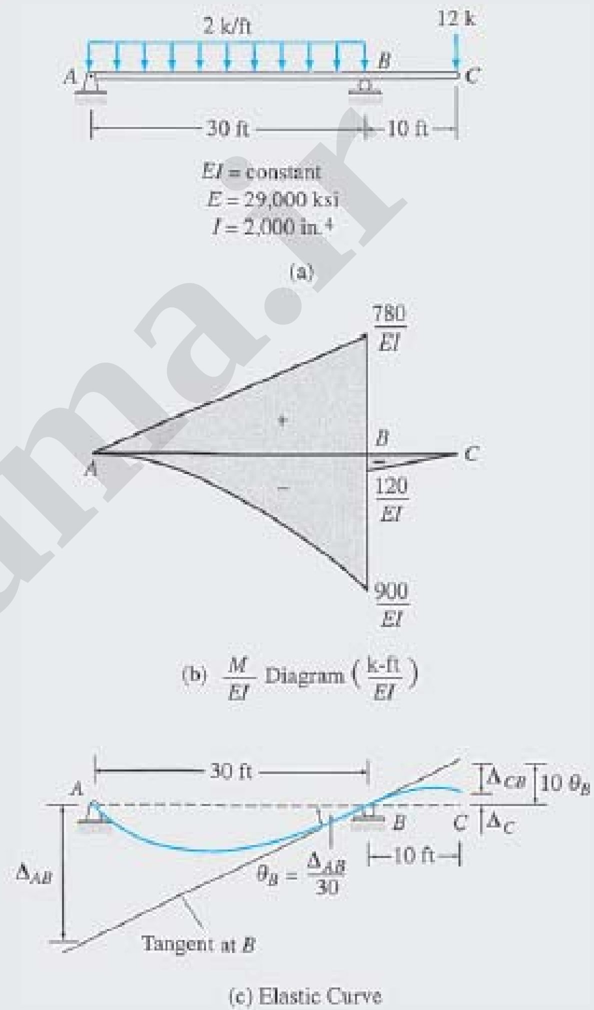


FIG. 6.11

Solution

M/EI Diagram. The bending moment diagram for this beam by cantilever parts with respect to the support point B was determined in Fig. 6.10. The ordinates of the bending moment diagram are divided by EI to obtain the M/EI diagram shown in Fig. 6.11(b).

Elastic Curve. See Fig. 6.11(c).

continued

Slope at *B*. Selecting the tangent at *B* as the reference tangent, it can be seen from Fig. 6.11(c) that

$$\theta_B = \frac{\Delta_{AB}}{30}$$

By using the M/EI diagram (Fig. 6.11(b)) and the properties of geometric shapes given in Appendix A, we compute

$$\Delta_{AB} = \frac{1}{EI}\left[\frac{1}{2}(780)(30)(20) - \frac{1}{3}(900)(30)\left(\frac{3}{4}\right)(30)\right]$$

$$= \frac{31{,}500 \text{ k-ft}^3}{EI}$$

Therefore,

$$\theta_B = \frac{31{,}500}{30EI} = \frac{1{,}050 \text{ k-ft}^2}{EI}$$

Deflection at *C*. From Fig. 6.11(c), we can see that

$$\Delta_C = 10\theta_B - \Delta_{CB}$$

where

$$\Delta_{CB} = \frac{1}{2}\left(\frac{120}{EI}\right)(10)\left(\frac{20}{3}\right) = \frac{4{,}000 \text{ k-ft}^3}{EI}$$

Therefore,

$$\Delta_C = 10\left(\frac{1{,}050}{EI}\right) - \frac{4{,}000}{EI} = \frac{6{,}500 \text{ k-ft}^3}{EI}$$

Substituting the numerical values of *E* and *I*, we obtain

$$\Delta_C = \frac{6{,}500(12)^3}{(29{,}000)(2{,}000)} = 0.194 \text{ in.}$$

$$\Delta_C = 0.194 \text{ in.} \uparrow$$ **Ans.**